

STRICT ∞ -CATEGORIES. CONCRETE DUALITY

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ABSTRACT. An elementary theory of strict ∞ -categories with application to concrete duality is given. New examples of first and second order concrete duality are presented.

1. Categories, functors, natural transformations, modifications

There are two kinds of weakness happenning to ∞ -categories. One is changing all occurrences of equality $=$ with a weaker equivalence relation \sim . The other one is a weak naturality condition. The first one is not proper and implies strict category theory. The second one is proper and gives a weak category theory. Below we use \sim instead of $=$. It is not necessary but has an advantage to treat directly the classification problem (up to \sim).

Definition 1.1.

- ∞ -precategory is a (big) set L endowed with
 - (1) grading $L = \coprod_{n \geq 0} L^n$
 - (2) unary operations $d, c : \coprod_{n \geq 1} L^n \rightarrow \coprod_{n \geq 0} L^n$, $\deg(d) = \deg(c) = -1$, $dc = d^2$, $cd = c^2$
 - (3) unary operation $e : \coprod_{n \geq 0} L^n \rightarrow \coprod_{n \geq 0} L^n$, $\deg(e) = 1$, $de = 1$, $ce = 1$
 - (4) partial binary operations \circ_k , $k = 1, 2, \dots$, of degree 0. $f \circ_k g$ is determined iff $d^k f = c^k g$ such that each **hom-set** $L(a, a') := \{f \in L \mid \exists k \in \mathbb{N} \, d^k f = a, \, c^k f = a'\}$, $\deg(a) = \deg(a')$, inherits all properties (1)-(4).
- $\forall a, a', a'' \in L^m$ there are maps $\mu_{a, a', a''} : \coprod_{n \geq 0} L^n(a', a'') \times L^n(a, a') \rightarrow L(a, a'')$ such that if the bottom composite is determined then

$$\begin{array}{ccc}
 \coprod_{n \geq 0} L^n(a', a'') \times L^n(a, a') & \xrightarrow{\mu_{a, a', a''}} & L(a, a'') \\
 \uparrow i \times i & & \uparrow i \\
 L^n(a', a'') \times L^n(a, a') & \xrightarrow{\circ_{n+1}} & L^n(a, a'')
 \end{array}$$

$\mu_{a, a', a''}$ are called **horizontal composites** on level $\deg(a)$, all composites inside of $L(a, a')$ are **vertical**. \square

Remarks.

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

- If $\alpha^n, \beta^n \in L^n, n > 0$, such that $d\alpha^n \neq d\beta^n$ or $c\alpha^n \neq c\beta^n$, then $L(\alpha^n, \beta^n) = \emptyset$ (because of $d^2 = dc, c^2 = cd$). So, $\mu_{a,a',a''}$ can be empty map $\emptyset : \emptyset \rightarrow \emptyset$.
- It is convenient to use a letter with appropriate superscript, like x^m, α^k , etc., as an element (or sometimes as a variable) with domain L^m, L^k , etc. respectively (or with domain $L^m(a, b), L^k(x, y)$, etc.) Also, grading can be taken over all \mathbb{Z} under assumption $L^{-m} := \emptyset, m > 0$.
- Call elements $a \in L^0$ of degree 0 by **objects** of L , elements $f^n \in L^n(a, a'), a, a' \in L^0$, by **arrows of degree $n + 1$ from a to a'** .
- Denote **horizontal composites** by $*$, and extend it over arrows of **different degrees** by the rule $*$: $L(b, c) \times L(a, b) \rightarrow L(a, c) : (g^n, f^m) \mapsto \mu_{a,b,c}(e^{\max(m,n)-n}g^n, e^{\max(m,n)-m}f^m) =: g^n * f^m$ ($f^m \in L^m(a, b), g^n \in L^n(b, c)$). \square

Definition 1.2. For $a, b \in L^n$ $a \sim b$ iff $\exists a \begin{smallmatrix} \xrightarrow{f} \\ \xleftarrow{g} \end{smallmatrix} b$ such that $e(a) \sim g \circ_1 f$ $f \circ_1 g \sim e(b)$

(it means that $\exists f \in L^0(a, b), g \in L^0(b, a)$ and two certain infinite sequences of arrows of higher order, one in $L(a, a)$ and the other in $L(b, b)$). \square

\sim is reflexive and symmetric, but may be not transitive.

Lemma 1.1. If L is an ∞ -precategory such that

\circ_1 is weakly associative $f \circ_1 (g \circ_1 h) \sim (f \circ_1 g) \circ_1 h$ (for composable arrows),

\circ_1 satisfies weak unit law $\forall f \in \coprod_{n \geq 1} L^n \begin{cases} f \circ_1 edf \sim f \\ ecf \circ_1 f \sim f \end{cases}$,

\sim is compatible with \circ_1 ($f \sim g$) & ($h \sim k$) $\Rightarrow (f \circ_1 h) \sim (g \circ_1 k)$ (for composable arrows),

\sim is transitive in higher orders, i.e. $\exists m > 0$ such that \sim is transitive for $\coprod_{n \geq m} L^n$,

then \sim is transitive in all orders.

Proof. Let $a \begin{smallmatrix} \xrightarrow{f} \\ \xleftarrow{g} \end{smallmatrix} b \begin{smallmatrix} \xrightarrow{f'} \\ \xleftarrow{g'} \end{smallmatrix} c$ be given equivalences, i.e. $ea \sim g \circ_1 f$, $eb \sim f \circ_1 g$, $eb \sim g' \circ_1 f'$,

$ec \sim f' \circ_1 g'$. Then $a \begin{smallmatrix} \xrightarrow{f' \circ_1 f} \\ \xleftarrow{g \circ_1 g'} \end{smallmatrix} c$ is the required equivalence since $ea \sim g \circ_1 f \sim g \circ_1 (eb \circ_1 f) \sim g \circ_1 ((g' \circ_1 f') \circ_1 f) \sim (g \circ_1 g') \circ_1 (f' \circ_1 f)$ and similarly $ec \sim (f' \circ_1 f) \circ_1 (g \circ_1 g')$. \square

Remarks.

- Transitivity in higher orders trivially holds for n -categories (starting from level n). For proper ∞ -categories it is better to make assumption ' \sim is transitive in all orders' from the beginning.
- This lemma shows that although transitivity of \sim is not automatic for ∞ -precategories, it is very consistent with (weak) associativity, unit law, and compatibility of \sim with composites. \square

Definition 1.3. ∞ -precategory L with relation \sim as above is called an ∞ -category iff

- \sim is transitive $\alpha \sim \beta \sim \gamma \Rightarrow \alpha \sim \gamma$,
- \sim is compatible with all composites $(f \sim g) \& (h \sim k) \Rightarrow (f \circ_n h) \sim (g \circ_n k)$ (when they are defined),
- horizontal composites preserve properties (1)-(2) and weakly preserve properties (3)-(4) of ∞ -precategories:
 - (1) grading $deg_{L(a,a'')}(\mu_{a,a',a''}(f, g)) = deg_{L(a',a'')}(f) = deg_{L(a,a')}(g)$
 - (2) $\mu_{a,a',a''}(df, dg) = d\mu_{a,a',a''}(f, g)$, $\mu_{a,a',a''}(cf, cg) = c\mu_{a,a',a''}(f, g)$
 - (3) $\mu_{a,a',a''}(ef, eg) \sim e\mu_{a,a',a''}(f, g)$

- (4) $\mu_{a,a',a''}(f \circ_k f', g \circ_k g') \sim \mu_{a,a',a''}(f, g) \circ_k \mu_{a,a',a''}(f', g')$ ("interchange law")
- each $\circ_k, k \in \mathbb{N}$, is **associative** $(f \circ_k g) \circ_k h \sim f \circ_k (g \circ_k h)$ (for composable elements),
 - **unit law** holds $e^k c^k f \circ_k f \sim f, f \circ_k e^k d^k f \sim f$ (when all operations are defined). \square

Remarks.

- By lemma 1.1, for n -categories transitivity condition on \sim follows from the others.
- Hom-sets in ∞ -category L are ∞ -categories themselves, and horizontal composites $*$: $L(b, c) \times L(a, b) \rightarrow L(a, c)$, are ∞ -functors.
- Since strict functors preserve equivalences \sim for categories in which horizontal composites preserve identity and composites strictly the compatibility condition on \sim with composites holds automatically. \square

A category is called **strict** if associativity and unit laws hold strictly up to $=$, and horizontal composites preserve identities and composites strictly. \sim still makes sense for strict categories.

Proposition 1.1. *In a strict ∞ -category L arrows of degree n (i.e., L^n) form 1-category with objects L^0 , arrows L^n , domain function d^n , codomain function c^n . $d, c : L^n \rightarrow L^{n-1}$ are 1-functors.* \square

Lemma 1.2.

- In ∞ -category L $e^k(f \circ_n g) \sim e^k f \circ_{n+k} e^k g$ (when either side is defined).
- \sim is preserved under \sim , i.e., if $a \xrightarrow{\sim_f} a'$ is an equivalence with $a' \xrightarrow{\sim_g} a$, its quasiinverse $(ea \sim g \circ f, ea' \sim f \circ g)$, and $f' \sim f$ then g is quasiinverse of f' as well.

- A quasiinverse is determined up to \sim , i.e. if $a \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \\ \xrightarrow{g'} \end{array} b$ and $g' \circ_1 f \sim ea \sim g \circ_1 f$ and $f \circ_1 g' \sim eb \sim f \circ_1 g$ then $g' \sim g$.

- All $n+1$ composites in $\mathbf{End}(e^n a) := L^0(e^n a, e^n a)$, $n \geq 0$ coincide up to equivalence \sim .

Proof.

- Assume $f, g \in L^m$, $m \geq n$, then $f \circ_n g = \mu_{d^n g, c^n f}(f, g)$ which weakly preserves e .
- $ea \sim g \circ f \sim g \circ f'$, $ea' \sim f \circ g \sim f' \circ g$.
- $g' = g' \circ_1 eb \sim g' \circ_1 f \circ_1 g \sim g \circ_1 f \circ_1 g \sim g \circ_1 eb = b$.
- $f \circ_{n+1} g = \mu_{a,a,a}(f, g) \sim \mu_{a,a,a}(f \circ_k e^{n+1} a, e^{n+1} a \circ_k g) \sim \mu_{a,a,a}(f, e^{n+1} a) \circ_k \mu_{a,a,a}(e^{n+1} a, g) \sim f \circ_k g$, $1 \leq k \leq n+1$. \square

Definition 1.4. An arrow $(f : a \rightarrow a') \in L^0(a, a')$, $\deg(a) = \deg(a') = m \geq 0$, is called

- **monic** if $\forall g, h : z \rightarrow a$ if $f \circ_1 g \sim f \circ_1 h$ then $g \sim h$
- **epic** if $\forall g', h' : a' \rightarrow w$ if $g' \circ_1 f \sim h' \circ_1 f$ then $g' \sim h'$
- **equivalence** if $\exists f' : a' \rightarrow a$ such that $edf \sim f' \circ_1 f$ and $edf' \sim f \circ_1 f'$ \square

Proposition 1.2. *For composable arrows*

- If f, g are monics then $f \circ_1 g$ is monic. If $f \circ_1 g$ is monic then g is monic
- If f, g are epics then $f \circ_1 g$ is epic. If $f \circ_1 g$ is epic then f is epic
- If f, g are equivalences then $f \circ_1 g$ is equivalence \square

Proposition 1.3. *All arrows representing equivalence $a \sim b$ are equivalences.* \square

Definition 1.5. ∞ -functor $F : L \rightarrow L'$ is a function which strictly preserves properties of precategories (1)-(2)

- (1) if $a \in L^n$ then $F(a) \in L'^n$

$$(2) F(da) = dF(a), F(ca) = cF(a)$$

and weakly preserves properties (3)-(4)

$$(3) F(ea) \sim eF(a)$$

$$(4) F(a \circ_k b) \sim F(a) \circ_k F(b)$$

□

Remarks.

- We do not require functor F to preserve equivalences \sim because it is not automatic and can be too restrictive. However, namely the functors preserving \sim are most important (e.g., see point 1.2).
- Inverse map F' for a bijective weak functor F is not a functor, in general. If F preserves \sim then for the inverse map F' to be a (weak) functor is equivalent to preserving \sim by F' . Inverse for a strict functor is always a strict functor. □

Lemma 1.3.

- *Strict functors preserve equivalences \sim .*
- *If functor $F : L \rightarrow L'$ is such that each its restriction on hom-sets $F_{a,b} : L(a, b) \rightarrow L'(F(a), F(b))$, $a, b \in L^0$, preserves equivalences \sim , then F preserves equivalences \sim .*
- *If $F : L \rightarrow L'$ is an embedding (injective map) such that $\forall a, b \in L^0 F_{a,b} : L(a, b) \rightarrow L'(F(a), F(b))$*

is a strict isomorphism and inverse F' to codomain restriction of $F : L \xrightarrow[F|_{Im(F)}]{F'} Im(F) \hookrightarrow L'$

is a functor then F reflects \sim .

Proof.

- Each arrow presenting a given equivalence $x \sim y$ is between a domain and a codomain which are constructed in a certain way only by composites and identity operations from arrows of smaller degree presenting the given equivalence and from elements x and y . A strict functor keeps the structure of the domains and codomains of arrows presenting equivalence $x \sim y$. So, the image of arrows presenting equivalence $x \sim y$ will be a family of arrows presenting equivalence $F(x) \sim F(y)$.

- For arrows of degree > 0 equivalences are preserved by assumption. Let $a \xrightleftharpoons[\sim]{f, g} b$, $a, b \in L^0$, be an equivalence for objects in L , i.e. $ea \sim g \circ_1 f$, $eb \sim f \circ_1 g$. Then there are two

opposite arrows $F(a) \xrightleftharpoons[\sim]{F(f), F(g)} F(b)$. By assumption, $F(ea) \sim F(g \circ_1 f)$, $F(eb) \sim F(f \circ_1 g)$. So,

$$eF(a) \sim F(ea) \sim F(g \circ_1 f) \sim F(g) \circ_1 F(f) \text{ and } eF(b) \sim F(eb) \sim F(f \circ_1 g) \sim F(f) \circ_1 F(g).$$

Therefore, $F(a) \xrightleftharpoons[\sim]{F(f), F(g)} F(b)$ is an equivalence.

- Inverse to a strict isomorphism is a strict isomorphism, i.e. preserves equivalences. So, F' is a functor which preserves equivalences in all hom-sets and, consequently, preserves all equivalences. Preservation of equivalences for F' is exactly reflection of equivalences for F . □

Lemma 1.4.

- $x = y$ iff $ex \sim ey$ [in particular, $=$ is definable via \sim].
- Functors, preserving \sim , strictly preserve all composites \circ_k , $k \geq 1$.
- Functors, weakly preserving e^2 , strictly preserve e , i.e. $e^2 F(a) \sim F(e^2 a) \Rightarrow eF(a) = F(ea)$.
- Quasiiequal functors (i.e. $F(f^n) \sim G(f^n)$ for all $f^n \in L^n$, $n \geq 0$) are equal.

Proof.

- $x = y \Rightarrow ex = ey \Rightarrow ex \sim ey$. Conversely, $ex \sim ey \Rightarrow dex = dey \Rightarrow x = y$.
- Sufficient to prove $eF(f \circ_k g) \sim e(F(f) \circ_k F(g))$, but it holds $eF(f \circ_k g) \sim F(e(f \circ_k g)) \sim (F \text{ preserves } \sim) F((ef) \circ_{k+1} (eg)) \sim F(ef) \circ_{k+1} F(eg) \sim eF(f) \circ_{k+1} eF(g) \sim e(F(f) \circ_k F(g))$.
- $e^2 F(a) \sim F(e^2 a) \Rightarrow de^2 F(a) = dF(e^2 a) \Rightarrow eF(a) = F(ea)$.
- Again, it is sufficient to prove $eF(f^n) \sim eG(f^n)$.
 $eF(f^n) \sim F(ef^n) \sim (\text{by assumption}) G(ef^n) \sim eG(f^n)$. \square

Corollary. ∞ -categories in the sense of definition 1.3 are almost strict, namely, with strict associativity, identity, and interchange laws.

Proof. Strict associativity and strict identity laws hold because by the axioms functors $L(x, y) \times L(y, z) \times L(z, t) \rightarrow L(x, t) : (f^n, g^n, h^n) \mapsto (h^n * g^n) * f^n$ and $L(x, y) \times L(y, z) \times L(z, t) \rightarrow L(x, t) : (f^n, g^n, h^n) \mapsto h^n * (g^n * f^n)$, $\deg(x) = \deg(y) = \deg(z) = \deg(t)$, are quasiequal, and, respectively, functors $L(x, y) \rightarrow L(x, y) : f \mapsto f$ and $L(x, y) \rightarrow L(x, y) : f \mapsto ey * f$, $\deg(x) = \deg(y)$ (similar for the right identity), are quasiequal. Strict interchange law is because functor $L(x, y) \times L(y, z) : (f, g) \mapsto g * f$ preserves \sim . \square

Definition 1.6. For given two functors F, G ∞ -natural transformation $\alpha : F \rightarrow G$ is a function $\alpha : L^0 \rightarrow L^1 : a \mapsto (F(a) \xrightarrow{\alpha(a)} G(a))$ such that

$$\mu_{F(a), F(b), G(b)}(e^k \alpha(b), F(f)) \sim \mu_{F(a), G(a), G(b)}(G(f), e^k \alpha(a))$$

for all $f \in L^k(a, b)$, $k = 0, 1, \dots$ \square

Definition 1.7. For given two functors F, G and two natural transformations $F \xrightleftharpoons[\beta]{\alpha} G$

∞ -modification $\lambda : \alpha \rightarrow \beta$ is a function $\lambda : L^0 \rightarrow L^2 : a \mapsto (\alpha(a) \xrightarrow{\lambda(a)} \beta(a))$ such that

$$\mu_{F(a), F(b), G(b)}(e^k \lambda(b), F(f)) \sim \mu_{F(a), G(a), G(b)}(G(f), e^k \lambda(a))$$

for all $f \in L^{k+1}(a, b)$, $k = 0, 1, \dots$ \square

Analogously, modifications of higher order are introduced. Call modifications by 1-modifications, natural transformations by 0-modifications.

Definition 1.8. Given two functors F, G , two 0-modifications $F \xrightleftharpoons[\alpha_2^0]{\alpha_1^0} G$,

two 1-modifications $\alpha_1^0 \xrightleftharpoons[\alpha_2^1]{\alpha_1^1} \alpha_2^0, \dots$, two $n-1$ -modifications $\alpha_1^{n-2} \xrightleftharpoons[\alpha_2^{n-1}]{\alpha_1^{n-1}} \alpha_2^{n-2}$

∞ - n -modification $\alpha^n : \alpha_1^{n-1} \rightarrow \alpha_2^{n-1}$ is a function $\alpha^n : L^0 \rightarrow L^{n+1} :$

$a \mapsto (\alpha_1^{n-1}(a) \xrightarrow{\alpha^n(a)} \alpha_2^{n-1}(a))$ such that

$$\mu_{F(a), F(b), G(b)}(e^k \alpha^n(b), F(f)) \sim \mu_{F(a), G(a), G(b)}(G(f), e^k \alpha^n(a))$$

for all $f \in L^{k+n}(a, b)$, $k = 0, 1, \dots$ \square

Corollary. All n -modification in the sense of definition 1.8 are strict.

Proof. By condition two functors $\alpha^n(b) * F(-) : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$ and $G(-) * \alpha^n(a) : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$ are quasiequal and, so, equal. \square

Definition 1.9. ∞ -**CAT** is an ∞ -category consisting of

- graded set $C = \coprod_{n \geq 0} C^n$, where C^0 are categories, C^1 functors, C^n ($n - 2$)-modifications
- if $\alpha^n : \alpha_1^{n-1} \rightarrow \alpha_2^{n-1} \in C^n$ then $d\alpha^n = \alpha_1^{n-1}$, $c\alpha^n = \alpha_2^{n-1}$
- $e\alpha^n \in C^{n+1}$ is the map $L^0 \rightarrow L'^{(n+1)} : a \mapsto e(\alpha^n(a))$
- for given two n -modifications α_1^n, α_2^n such that $d^k\alpha_1^n = c^k\alpha_2^n$

$$\alpha_1^n \circ_k \alpha_2^n := \begin{cases} a \mapsto (\alpha_1^n(a) \circ_k \alpha_2^n(a)) & \text{if } k < n + 2 \\ a \mapsto (\alpha_1^n(F'(a)) \circ_{(n+1)} G(\alpha_2^n(a))) & \text{if } k = n + 2, F' = c^{(n+1)}\alpha_2^n, G = d^{(n+1)}\alpha_1^n \end{cases}$$

First composite works when $\alpha_1^n, \alpha_2^n \in \infty\text{-CAT}(L, L')$, second when $\alpha_1^n \in \infty\text{-CAT}(L', L'')$ $\alpha_2^n \in \infty\text{-CAT}(L, L')$, where L, L', L'' are categories. \square

Proposition 1.4. *Categories, functors, natural transformations, modifications, etc. constitute ∞ -category $\infty\text{-CAT}$ of ∞ -categories.* \square

Definition 1.10. Category L is called ∞ - n -category if $L^{j+1} = e(L^j)$ for $j \geq n$. \square

L/\sim is not a category in general since \sim is not compatible with e . However, if we take quotient only on a fixed level n and make all higher arrows identities we get ∞ - n -category $L^{(n)}$, n -th approximation of L . Generally there are no functors $L^{(n)} \xrightarrow{\hookrightarrow} L$, $L \xrightarrow{\twoheadrightarrow} L^{(n)}$ (except for the last surjection if L is a weak ∞ -($n + 1$)-category and all $(n + 1)$ -arrows are iso's).

1.a. Weak categories, functors, natural transformations, modifications.

As we saw above, using a weak language (substitution \sim instead of $=$) does not give a weak category theory. The only advantage was that we could deal with \sim instead of $=$ (which is important for the classification problem that still makes sense for strict ∞ -categories). All known definitions of weak categories [C-L, Lei, Koc, etc.] are nonelementary (at least, they use functors, natural transformations, operads, monads just for the very definition). Probably, this is a fundamental feature of weak categories. To introduce them we also need the whole universe $\infty\text{-PreCat}$ of ∞ -precategories.

Definition 1.a.1. $\infty\text{-PreCat}$ consists of

- $\infty\text{-precategories}$ (definition 1.1) together with \sim -relation in each [\sim may be not transitive],
- $\infty\text{-functors}$ (definition is like 1.5 for ∞ -categories), i.e. functions $F : L \rightarrow L'$ of degree 0 preserving d and c strictly, and e and \circ_k , $k \geq 1$, weakly,
- **lax ∞ - n -modifications**, $n \geq 0$, i.e. **total** maps $\alpha^n : L \rightarrow L'$ (with variable degree on different elements, but $\leq n + 1$, more precisely, the induced map $\mathbb{N} \rightarrow \mathbb{N} : \deg(x) \mapsto (\deg(\alpha^n(x)) - \deg(x))$ is an antimonotone map, decreasing by 1 at each step from $n + 1$ at $\deg(x) = 0$ to 1 at $\deg(x) = n$ and remaining constant 1 after) being defined for a given sequence of two functors $F, G : L \rightarrow L'$, two 0-modifications (natural transformations) $\alpha_1^0, \alpha_2^0 : F \rightarrow G$, ..., two $(n - 1)$ -modifications $\alpha_1^{n-1}, \alpha_2^{n-1} : \alpha_1^{n-2} \rightarrow \alpha_2^{n-2}$ as $\alpha^n :=$

$$\left\{ \begin{array}{ll}
(\alpha^n(x) : \alpha_1^{n-1}(x) \rightarrow \alpha_2^{n-1}(x)) \in L'^n(F(x), G(x)) & x \in L^0 \\
\alpha^n(x) := \alpha^n(e^{n+1-k}x) \in L'^{n+1}(F(d^k x), G(c^k x)) & x \in L^k \\
& 0 < k < n+1 \\
(\alpha^n(x) : \alpha^n(c^{n+1}x) \circ_{n+1} F(x) \rightarrow G(x) \circ_{n+1} \alpha^n(d^{n+1}x)) \in & x \in L^{n+1} \\
& \in L'^{n+1}(F(d^{n+1}x), G(c^{n+1}x)) \\
(\alpha^n(x) : \alpha^n(cx) \circ_1 (e\alpha^n(c^{n+2}x) \circ_{n+2} F(x)) \rightarrow & x \in L^{n+2} \\
(G(x) \circ_{n+2} e\alpha^n(d^{n+2}x)) \circ_1 \alpha^n(dx)) \in L'^{n+2}(F(d^{n+2}x), G(c^{n+2}x)) \\
\alpha^n(x) : \alpha^n(cx) \circ_1 (e\alpha^n(c^2x) \circ_2 (e^2\alpha^n(c^{n+3}x) \circ_{n+3} F(x))) \rightarrow & x \in L^{n+3} \\
((G(x) \circ_{n+3} e^2\alpha^n(d^{n+3}x)) \circ_2 e\alpha^n(d^2x)) \circ_1 \alpha^n(dx) \in L'^{n+3}(F(d^{n+3}x), G(c^{n+3}x)) \\
\vdots \\
\alpha^n(x) : & x \in L^{n+m} \\
\alpha^n(cx) \circ_1 \cdots \circ_{m-2} (e^{m-2}\alpha^n(c^{m-1}x) \circ_{m-1} (e^{m-1}\alpha^n(c^{n+m}x) \circ_{n+m} F(x)) \underbrace{\cdots}_{m-1}) \rightarrow \\
\underbrace{(\cdots (G(x) \circ_{n+m} e^{m-1}\alpha^n(d^{n+m}x)) \circ_{m-1} e^{m-2}\alpha^n(d^{m-1}x)) \circ_{m-2} \cdots \circ_1 \alpha^n(dx)}_{m-1} \in \\
& \in L'^{n+m}(F(d^{n+m}x), G(c^{n+m}x)) \\
\vdots \\
d\alpha^n := \alpha_1^{n-1}, \quad c\alpha^n := \alpha_2^{n-1} \quad [(d\alpha^n)(x) \neq d(\alpha^n(x)), (c\alpha^n)(x) \neq c(\alpha^n(x)) \text{ if } \deg(x) > 0]. \quad \square
\end{array} \right.$$

Remarks.

- ∞ - n -modifications look terrible but it is the weakest form of naturality (infinite sequence of naturality squares for naturality squares). To deal with such entities a kind of operads is needed.
- To give n -modification α^n is the same as to give a map $\alpha^n|_{L^0} : L^0 \rightarrow L'$ of degree $n+1$ and $\forall a, b \in L^0$ a natural transformation $\nu_{a,b}^{\alpha^n} : \alpha^n(b) * F(-) \rightarrow G(-) * \alpha^n(a) : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$, where $F = d^{n+1}\alpha^n$, $G = c^{n+1}\alpha^n$.
- When $\alpha^n(x)$, $\deg(x) > 0$, are all identities (of the required types) ∞ - n -modifications are called **strict**. They are usual modifications and composable like in definition 1.9 when universe ∞ -**CAT** is strict (in that case strict modifications are weak as well). In a weak universe ∞ -**CAT** strict modifications need not to be weak (i.e. to be modifications at all).
- ∞ -**PreCAT** is not an ∞ -precategory itself because there are no identities and composites for weak n -modifications. The problem with identities and composites is not clear, if they exist at all without making either naturality condition or ∞ -categories stricter.
- In general, these two sides 'categories and functors' and ' n -modifications' form a strange pair. If we weaken one of these sides the other one becomes stricter (under condition that ∞ -**CAT** is a (let it be very weak) **category**). So, the following **hypothesis** holds:

*There is no ∞ -**CAT** with simultaneously weak categories, functors, and n -modification.*

For example, if we want weak modifications and want them to be composable we need to introduce several axioms on categories, one of which is like ' $\forall a, b \in L^0$ and \forall functors $F, G : L \rightarrow L'$ if \exists natural transformations $\alpha : f_1 * F(-) \rightarrow G(-) * g_1 : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$ and $\beta : f_2 * F(-) \rightarrow G(-) * g_2 : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$ and $n+1$ -cells f_1, f_2 and g_1, g_2 are \circ_k -composable then \exists a natural transformation (k -composite) $\gamma : (f_1 \circ_k f_2) * F(-) \rightarrow G(-) * (g_1 \circ_k g_2) : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$ '. But such axioms make categories very special. From the other side if we want categories to be weak we need to make stricter (maybe,

strict) n -modifications in order they would be composable. The problem is in existence of composites (and units) for weak n -modifications.

- Instead of lax n -modifications we could use modifications with $\alpha^n(x)$ being \sim for $\deg(x) > 0$ in L' . In both cases in order to make horizontal composites (at least, $F * \alpha^n := F \circ_{\mathbf{SET}} \alpha^n$) we need functors preserving composites (or composites and \sim), i.e. 'weak modifications' \Rightarrow 'strict functors'.
- If the above hypothesis was true it could be nice, e.g. a universe where ∞ -**Top** lives would contain only strict n -modifications. \square

Definition 1.a.2. Weak ∞ -category L is an ∞ -precategory (see definition 1.1) such that [all elements below are supposed to be composable when we write composite for them]

- \sim is transitive $x \sim y \sim z \Rightarrow x \sim z$,
- horizontal composites $*$ strictly preserve properties (1)-(2) of precategories
 - (1) $\deg(x * y) = \deg(x) = \deg(y)$ if $\deg(x) = \deg(y)$ (interchange law for degree)
 - (2) $d(x * y) = (dx) * (dy)$, $c(x * y) = (cx) * (cy)$ if $\deg(x) = \deg(y)$ (interchange law for domain and codomain)
 and weakly preserve properties (3)-(4) of precategories
 - (3) $e(x * y) \sim (ex) * (ey)$ if $\deg(x) = \deg(y)$ (interchange law for identity)
 - (4) $(x \circ_k y) * (z \circ_k t) \sim (x * z) \circ_k (y * t)$ if $\deg(x) = \deg(y) = \deg(z) = \deg(t)$ (interchange law for composites) [\circ_k has smaller 'deepness' k than the given $*$ $= \circ_n$, $n > k$],
- **(weak associativity)**
 $\forall x, y, z, t \in L^n$ for two functors $l_{x,y,z,t} : L(x, y) \times L(y, z) \times L(z, t) \rightarrow L(x, t) : (f, g, h) \mapsto (h * g) * f$ and $r_{x,y,z,t} : L(x, y) \times L(y, z) \times L(z, t) \rightarrow L(x, t) : (f, g, h) \mapsto h * (g * f) \exists$ natural transformation $\alpha_{x,y,z,t} : l_{x,y,z,t} \rightarrow r_{x,y,z,t}$,
- **(weak unit)**
 $\forall x, y \in L^n$ and functors $u_{x,y}^l : L(x, y) \rightarrow L(x, y) : f \mapsto ey * f$ and $u_{x,y}^r : L(x, y) \rightarrow L(x, y) : f \mapsto f * ex \exists$ natural transformations $\epsilon_{x,y}^l : u_{x,y}^l \rightarrow Id$ and $\epsilon_{x,y}^r : Id \rightarrow u_{x,y}^r$. \square

Remarks.

- We do not introduce a universe ∞ -**CAT** with weak categories, functors and n -modifications because there is no (at least, obvious) units and composites for n -modifications (however, identity natural transformations exist if only vertical composites of natural transformations are defined, for if $F : L \rightarrow L'$ is a functor take $(eF)(a) := e(F(a))$, $a \in L^0$ and by the weak unit law $\forall a, b \in L^0 \exists$ a natural transformation $\nu_{a,b} : e(F(b)) * F(-) \rightarrow F(-) * e(F(a)) : L^{\geq 0}(a, b) \rightarrow L'^{\geq 0}(F(a), F(b))$, take $\nu_{a,b} := (\epsilon_{F(a), F(b)}^{u^r} \circ_1 \epsilon_{F(a), F(b)}^{u^l}) * F := (\epsilon_{F(a), F(b)}^{u^r} \circ_1 \epsilon_{F(a), F(b)}^{u^l}) \circ_{\mathbf{SET}} F$. The problem is what are the weakest conditions on categories, functors and n -modifications in order they form a category. Maybe, there are several independent such conditions and, so, several categories living in ∞ -**CAT** with weakest entities.
- To keep a usual form of (weak) associativity and (weak) unit we could introduce relations \sim_k for elements of images of two functors $F, G : L \rightarrow L'$ connected by a natural transformation $\alpha : F \rightarrow G$, namely, $x \sim_k y$ if $\exists z \in L^k$ such that $x = F(z)$, $y = G(z)$. These relations are not reflexive, symmetric and transitive. Then we could write associativity and unit laws as $(x \circ_k y) \circ_k z \sim_{k-1} x \circ_k (y \circ_k z)$ and $e^k c^k x \circ_k x \sim_{k-1} x$, $x \sim_{k-1} x \circ_k e^k d^k x$. Under assumption that composites and units exist in an ∞ -**CAT** we could choose more sensible piece of ∞ -**CAT** with categories in which $\sim_0 \equiv \sim$ and all \sim_k are symmetric and transitive by the requirement that $\alpha_{x,y,z,t}$, $\epsilon_{x,y}^l$, $\epsilon_{x,y}^r$ are equivalences. \square

Examples

1. ∞ -**Top** is an ∞ -category with homotopies for homotopies as higher order cells.
2. ∞ -**Diff** is an ∞ -category of differentiable manifolds in the same way as ∞ -**Top**.
3. ∞ -**TopAlg** is an ∞ -category of topological algebras in the same way as ∞ -**Top** where each instance of homotopy is a homomorphism of topological algebras.
4. **2-Top** is a strict ∞ -2-category with 2-cells, homotopy classes of homotopies, and just identities in higher order (\sim on the level of objects means homotopy equivalence of spaces, on the level of 1-arrows homotopness of maps, and on the level ≥ 2 coincidence). **2-Cat** is similar.
5. ∞ -**Compl** is an ∞ -category of (co)chain complexes with (algebraic) homotopies for homotopies as higher order cells.
6. For 1-category A , A_{equiv} is a strict ∞ -2-category such that $A_{equiv}^0 = A^0$, $A_{equiv}^1 = \left\{ f \in \right.$

$$A \left| \begin{array}{ccc} \bullet & \xrightarrow{\exists H} & \bullet \\ \uparrow f & \sim & \uparrow f \\ \bullet & \xrightarrow{\forall h} & \bullet \end{array} \right\}, A_{equiv}^2 = \left\{ \text{iso's} \mid \forall f, g \in A_{equiv}^1 \exists! f \xrightarrow{\sim} g \text{ iff } \begin{array}{ccc} \bullet & \xrightarrow{\exists H} & \bullet \\ \uparrow f & \sim & \uparrow g \\ \bullet & \xrightarrow{\forall h} & \bullet \end{array} \right\}.$$

A_{equiv} contains all equivariant maps $f : X \rightarrow Y$ with respect to a group homomorphism $\rho : \mathbf{Aut}(X) \rightarrow \mathbf{Aut}(Y)$.

7. (weak) **covariant ∞ -Hom-functor** $L(a, -) : L \rightarrow \infty\text{-CAT}$:

$$\left\{ \begin{array}{ll} b \mapsto L(a, b) & b \in L^0 \\ (f : b \rightarrow b') \mapsto (L(a, f) : g \mapsto \mu(e^k f, g)) & f \in L^0(b, b'), g \in L^k(a, b) \\ (\alpha : f \rightarrow f') \mapsto (L(a, \alpha) : x \mapsto \mu(\alpha, ex)) & \alpha \in L^1(b, b'), x \in L^0(a, b) \\ (\delta : \alpha \rightarrow \alpha') \mapsto (L(a, \delta) : x \mapsto \mu(\delta, e^2 x)) & \delta \in L^2(b, b'), x \in L^0(a, b) \\ \dots & \\ (\alpha^n : \alpha_1^{(n-1)} \rightarrow \alpha_2^{(n-1)}) \mapsto (L(a, \alpha^n) : x \mapsto \mu(\alpha^n, e^n x)) & \alpha^n \in L^n(b, b'), x \in L^0(a, b) \\ \dots & \end{array} \right.$$

8. **opposite category** L^{op} is an ∞ -category such that

- $(L^{op})^n = L^n$, $n \geq 0$
- $d^{op}(\alpha^n) = \begin{cases} d(\alpha^n) & \text{if } n \geq 2 \\ c(\alpha^n) & \text{if } n = 1 \end{cases} \quad c^{op}(\alpha^n) = \begin{cases} c(\alpha^n) & \text{if } n \geq 2 \\ d(\alpha^n) & \text{if } n = 1 \end{cases}$
- $e^{op} = e$
- $\beta^n \circ_k^{op} \alpha^n = \begin{cases} \beta^n \circ_k \alpha^n & \text{if } \alpha^n, \beta^n \in L^n, k < n \\ \alpha^n \circ_k \beta^n & \text{if } \alpha^n, \beta^n \in L^n, k = n \end{cases} \quad (\text{for composable elements})$

9. (weak) **contravariant ∞ -Hom-functor** $L(-, b) : L^{op} \rightarrow \infty\text{-CAT}$:

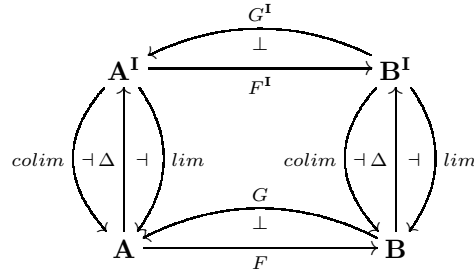
$$\left\{ \begin{array}{ll} a \mapsto L(a, b) & a \in L^0 \\ (f : a \rightarrow a') \mapsto (L(f, b) : g \mapsto \mu(g, e^k f)) & f \in L^0(a, a'), g \in L^k(a', b) \\ (\alpha : f \rightarrow f') \mapsto (L(\alpha, b) : x \mapsto \mu(ex, \alpha)) & \alpha \in L^1(a, a'), x \in L^0(a', b) \\ (\delta : \alpha \rightarrow \alpha') \mapsto (L(\delta, b) : x \mapsto \mu(e^2 x, \delta)) & \delta \in L^2(a, a'), x \in L^0(a', b) \\ \dots & \\ (\alpha^n : \alpha_1^{(n-1)} \rightarrow \alpha_2^{(n-1)}) \mapsto (L(\alpha^n, b) : x \mapsto \mu(e^n x, \alpha^n)) & \alpha^n \in L^n(a, a'), x \in L^0(a', b) \\ \dots & \end{array} \right.$$

10. **Yoneda embedding** $\mathbf{Y} : L \rightarrow \infty\text{-CAT}^{L^{op}} : \alpha \mapsto L(-, \alpha)$, $\alpha \in L$.

11. **Set** is simultaneously an object and a full subcategory of $\infty\text{-CAT}$.

12. A (big) set $L_{\sim} := \coprod_{n \geq 0} L_{\sim}^n$, where L_{\sim}^n are defined recursively as $L_{\sim}^0 := L^0$ and L_{\sim}^n are all equivalences from L^n with domain and codomain in L_{\sim}^{n-1} , is a subcategory of L . Similarly, $L_{k\sim} := \coprod_{n \geq 0} L_{k\sim}^n$, $k \geq 0$, where $L_{k\sim}^n := \begin{cases} L^n & n \leq k \\ \text{equivalences from } L^n \text{ with dom and codom in } L_{k\sim}^{n-1} & n > k \end{cases}$, is a subcategory of L . From this point $L_{\sim} = L_{0\sim}$. Such categories are most important for the classification problem (up to \sim). Sometimes, 'invariants' can be constructed only for L_{\sim} (see point 2.1).
13. **Higher order concepts** can **simplify** proof of first order facts. E.g., each strict 2-functor $\Phi : 2\text{-}\mathbf{CAT} \rightarrow 2\text{-}\mathbf{CAT}$, where $2\text{-}\mathbf{CAT}$ is the usual strict category of categories, functors, and natural transformations, preserves adjunction (indeed, triangle identities $\begin{cases} \varepsilon G \circ G \eta = 1_G \\ F \varepsilon \circ \eta F = 1_F \end{cases}$ are respected by Φ $\begin{cases} \Phi(\varepsilon)\Phi(G) \circ \Phi(G)\Phi(\eta) = 1_{\Phi(G)} \\ \Phi(F)\Phi(\varepsilon) \circ \Phi(\eta)\Phi(F) = 1_{\Phi(F)} \end{cases}$). It gives short proofs of the following results.
- a) *Right adjoints preserve limits (left adjoints preserve colimits).*

PROOF.



where $(-)^I \equiv 2\text{-}\mathbf{CAT}(\mathbf{I}, -) : 2\text{-}\mathbf{CAT} \rightarrow 2\text{-}\mathbf{CAT}$ is a hom-2-functor.

Now, $G^I \circ \Delta = \Delta \circ G$ (obvious). Taking right adjoints of both sides completes the proof $\lim \circ F^I \simeq F \circ \lim$ (for colimits the same argument works $F^I \circ \Delta = \Delta \circ F \Rightarrow \text{colim} \circ G^I \simeq G \circ \text{colim}$). \square

- b) *Each 1-Cat-valued presheaf admits a sheafification (1-Cat is a category of small categories and functors between them).*

PROOF. 1-Cat-valued presheaf on \mathbf{C} is the same as an internal category object in $\mathbf{Set}^{\mathbf{C}^{op}}$.

There is an adjoint situation $\mathbf{Sh}(\mathbf{C}) \xrightleftharpoons{\perp} \mathbf{Set}^{\mathbf{C}^{op}}$ in \mathbf{LEX} , where $\mathbf{LEX} \hookrightarrow 2\text{-}\mathbf{CAT}$ is a 2-category of finitely complete categories, functors preserving finite limits, and (arbitrary) natural transformations. There is a 2-functor $\mathbf{CAT}(-) : \mathbf{LEX} \rightarrow 2\text{-}\mathbf{CAT}$ assigning to each category in \mathbf{LEX} the category of its internal category objects and to each functor and natural transformation the induced ones. Then \exists an adjunction $\mathbf{CAT}(\mathbf{Sh}(\mathbf{C})) \xrightleftharpoons{\perp} \mathbf{CAT}(\mathbf{Set}^{\mathbf{C}^{op}})$

which means that each 1-Cat-valued presheaf can be sheafified by the top curved arrow. \square

1.1. Fractal organization of the new universum.

Fractal Principle. Object A with properties $\{P_i\}_I$ has fractal structure if there are subobjects $\{A_j\}_J$ which relate to each other in a certain way (express it by additional property $P =$ 'to have $|J|$ subobjects which relate in the certain way') and each A_j inherits all properties $\{P_i\}_I \& P$. \square

It can be useful to see that in spite of complicated structure each ∞ -category and, moreover, $\infty\text{-}\mathbf{CAT}$ has a regular structure which is repeated for certain arbitrary small pieces. Such pieces are, of course, hom-sets $L(a, b)$ which inherit all properties (1)-(4), associativity and identity laws, and each piece of which still has the same structure. In particular, $L(a, b)(c, d) = L(c, d)$. ∞ -

functor restricted to such a piece is again ∞ -functor. Moreover, each ∞ -category can be regarded as a hom-set of a little bit bigger category if we formally attach two distinct elements $\alpha, \beta \in L^{-1}$ with their identities of higher order $e^n(\alpha), e^n(\beta)$, $n \geq 1$ (such that $d(L^0) = \alpha$, $c(L^0) = \beta$ and composites with these identities of other elements hold strictly). Other natural pieces of L which inherit all properties and are ∞ -categories are $L^{\geq n}$, $L^{\geq n}(a, b)$ (elements of degree not lower than n).

1.2. Notes on Coherence Principle.

This principle is an axiom to deal with equivalence relation \sim . It is not logically necessary for higher order category theory itself. There can be categories in which it does not hold.

Coherence Principle. For a given set of cells $\{a_i\}_I$ and a given set of base equivalences $\{t_j(\{a_i\}_I) \sim s_j(\{a_i\}_I)\}_J$ for any two constructions $F_1(\{a_i\}_I)$ and $F_2(\{a_i\}_I)$ and any two derived equivalences $\varepsilon_i^0 : F_1(\{a_i\}_I) \sim F_2(\{a_i\}_I)$, $i = 1, 2$ there are derived equivalences $\varepsilon_m^1 : \varepsilon_1^0 \sim \varepsilon_2^0$, $m \in M^1$, such that for any two of them $\varepsilon_{m_1}^1, \varepsilon_{m_2}^1$ there are derived equivalences $\varepsilon_m^2 : \varepsilon_{m_1}^1 \sim \varepsilon_{m_2}^1$, $m \in M^2$ again such that for any pair of them $\varepsilon_{m_1}^2, \varepsilon_{m_2}^2$ there are derived equivalences of higher order, etc. \square

Here constructions mean application of composites, functors, natural transformations,.. to $\{a_i\}_I$. Derived equivalences mean equivalences obtained from base ones by virtue of categorical axioms.

2. (m, n) -invariants

Definition 2.1.

- **Equivalence** $x^k \sim y^k$, $x^k, y^k \in L^k$, $k \geq 0$, is called **of degree** l , $\deg(\sim) := l$, $l \geq 0$, if all arrows representing it starting from order $k + l + 1$ and higher are identities and for $l > 0$ there is at least one nonidentity arrow on level $k + l$. If there is no such $l \in \mathbb{N}$, $\deg(\sim) := \infty$. Denote \sim of degree l by \sim_l .
- **Pair of equivalent elements** $x^k \sim y^k$, $k \geq 0$, is called **of degree** l , $\deg(x^k \sim y^k) := l$, $l \geq 0$, if the lowest degree of equivalences existing between x^k and y^k is l .
- **∞ -category** L is called **of degree** l , $\deg(L) = l$, $l \geq 0$, if for any pair of equivalent objects $a \sim a'$, $a, a' \in L^0$, there exists an equivalence $a \sim_k a'$ of degree $k \leq l$ and there exists at least one pair of equivalent objects from L of degree l .
- **Functor** $F : L \rightarrow L'$ is called **(m, n) -invariant** if F preserves equivalences \sim , $m = \deg(L)$, $0 \leq n \leq \deg(L')$ and F maps every pair of equivalent objects of degree $\leq m$ to a pair of equivalent objects of degree $\leq n$, i.e. $\deg(a \sim a') \leq m \Rightarrow \deg(F(a) \sim F(a')) \leq n$, and boundary n is actually achieved on a pair of equivalent objects of L . \square

Remarks.

- (m, n) -invariants are important for the classification problem (up to \sim). If $n < m$ an (m, n) -invariant decreases complexity of the equivalence relation, i.e. partially resolves it.
- There can be trivial invariants which do not distinguish anything and do not carry any information such as constant functors $c : L \rightarrow L'$ (although they are $(\deg(L), 0)$ -invariants). \square

Examples

1. $\deg(ea) = 0$; $\deg(f : a \xrightarrow[iso]{\sim} a') = 1$; $\deg(\mathbf{Set}) = 1$; $\deg(\infty\mathbf{-Top}) = 2$; $\deg(\infty\mathbf{-CAT}) = \infty(?)$.
2. Homology and cohomology functors $H_*, H^* : \infty\mathbf{-Top} \rightarrow \mathbf{Ab}$ (trivially extended over higher order cells) are $(2, 1)$ -invariants.
3. $\pi_n^I / \sim : L_{1\sim}^* \rightarrow \mathbf{Grp}$ is an $(\infty, 1)$ -invariant (see proposition 2.1.2).

4. Let X be a smooth manifold with Lie group action $\rho : G \times X \rightarrow X$, L be a category with L^0 , the set of submanifolds of X , $L^1(a, b) := \{(a, g, b) \in L^0 \times G \times L^0 \mid \rho(g, a) = b\}$, $L^n := eL^{n-1}$ for $n \geq 2$, L' be a category with $L'^0 := C^\infty(X, \mathbb{R})$ (smooth functions), $L'^1(f, h) := \{(f, g, h) \in L'^0 \times G \times L'^0 \mid f \circ \rho(g^{-1}, -) = h\}$, $L'^n := eL'^{n-1}$ for $n \geq 2$. If $F : L \rightarrow L'$ is a construction (functor) assigning invariant functions to objects from L then F is an $(1, 0)$ -invariant.
5. Each equivalence $L \xrightarrow{\sim} L'$ is $(\deg(L), \deg(L'))$ -invariant with $\deg(L) = \deg(L')$.

2.1. Homotopy groups associated to ∞ -categories.

Let L be an ∞ -category in which $*$ strictly preserves e and \sim (i.e. $*$ is a strict functor). Denote by $eqL := \{f \in L \mid \exists g. edf \sim g \circ_1 f, edg \sim f \circ_1 g\}$ subset of equivalences of ∞ -category L . It can be not a category (because it is not closed under d, c , in general).

Definition 2.1.1. Assume, $L(I, -) : L \rightarrow \infty\text{-CAT}$, $x \in L^0(I, a)$. Then $\pi_n^I(a, x) :=$

$$\begin{cases} (L^0(I, a), x) & \text{if } n=0 \\ \mathbf{Aut}_{L(I, a)}(e^{n-1}x) := eqL(I, a)(e^{n-1}x, e^{n-1}x) \cap (L(I, a))^0(e^{n-1}x, e^{n-1}x) = \\ = eqL(e^{n-1}x, e^{n-1}x) \cap L^{n+1} & \text{if } n > 0 \end{cases}$$

are (weak) **homotopy groups** of object a at point x with representing object $I \in L^0$. \square

$\pi_0^I(a, x)$ or $\pi_0^I(a, x)/\sim$ are just pointed sets, $\pi_n^I(a, x)/\sim, n > 0$ are strict groups.

Remark. If $L = \infty\text{-Top}$, $I = \mathbf{1}$ then the above homotopy groups are usual ones. \square

Definition 2.1.2. For a map $f : a \rightarrow b$ such that $f \circ x = y$, $x \in L^0(I, a)$, $y \in L^0(I, b)$ the **induced map** $f_* \equiv \pi_n^I(f) : \pi_n^I(a, x) \rightarrow \pi_n^I(b, y)$ is determined by restriction of functor $L(I, f) :$

$$\begin{cases} L^0(I, a) \rightarrow L^0(I, b) : x' \mapsto f \circ_1 x' & \text{if } n = 0 \\ \mathbf{Aut}_{L(I, a)}(e^{n-1}x) \rightarrow \mathbf{Aut}_{L(I, b)}(e^{n-1}y) : g \mapsto \mu_{I, a, b}(e^n f, g) & \text{if } n > 0 \end{cases} \quad \square$$

Remark. To be correctly defined induced maps $\pi_n^I(f)$ for $n > 1$ need commutativity of $*$ with e . First two 'groups' $\pi_0^I(a, x), \pi_1^I(a, x)$ always make sense and depend functorially on objects. \square

Proposition 2.1.1 (homotopy invariance of homotopy groups). *If $x : I \rightarrow a$, $f \sim f' \in L^0(a, b)$ such that $f \circ_1 x \sim f' \circ_1 x$ is trivial equivalence (all arrows for \sim are identities) then $\pi_n^I(f)/\sim = \pi_n^I(f')/\sim : \pi_n^I(a, x)/\sim \rightarrow \pi_n^I(b, f \circ x)/\sim$.*

Proof is immediate. \square

Proposition 2.1.2. $\pi_n^I/\sim : L_{1\sim}^* \rightarrow \mathbf{Grp}$ is an $(\infty, 1)$ -invariant, where $L_{1\sim}^* := \coprod_{n \geq 0} L_{1\sim}^{*n}$, $L_{1\sim}^{*n} :=$

$$\begin{cases} L^{*n} \text{ (pointed objects and maps)} & n = 0, 1 \\ \text{equivalences from } L^n \text{ with dom and codom in } L_{1\sim}^{*(n-1)} & n > 1 \end{cases}.$$

Proof. Partial functor $\pi_n^I/\sim : L^{*0} \coprod L^{*1} \rightarrow \mathbf{Grp}$ is trivially extendable starting from equivalences on level 2 (because of proposition 2.1.1). \square

Example (Fundamental Group)

Let 2-Top be usual \mathbf{Top} with homotopy classes of homotopies as 2-cells. Define **fundamental groupoid** 2-functor as representable $\Pi(-) := Hom_{2\text{-Top}}(\mathbf{1}, -) : 2\text{-Top} \rightarrow 2\text{-Cat}$:

$$\left\{ \begin{array}{ll} X \rightarrow \Pi(X) & Ob(\Pi(X)) \text{ are its points, } Ar(\Pi(X)) \text{ are homotopy classes of pathes} \\ (X \xrightarrow{f} Y) \mapsto \Pi(f) & \text{transformation of fundamental groupoids, } \Pi(f) : \begin{cases} x \mapsto f(x) \\ [\gamma] \mapsto [f \circ \gamma] \end{cases} \\ (f \xrightarrow{[H]} f') \mapsto \Pi([H]) & \text{nat. trans. } \Pi([H]) = \{[H] * i_x\}_{x \in X} : Hom_{2\text{-}\mathbf{Top}}(1, f) \xrightarrow{\sim} Hom_{2\text{-}\mathbf{Top}}(1, f') \\ \text{(where } \{[H] * i_x\}_{x \in X} = \{[H(x, -)]\}_{x \in X} \text{ are homotopy classes of pathes between } f(x) \text{ and } f'(x) \text{ natural in } x \in X\text{).} \end{array} \right.$$

$\pi_1(X, x_0) := \mathbf{Aut}_{\Pi(X)}(x_0) \hookrightarrow \Pi(X)$ is **fundamental group** of space X at point $x_0 \in X$,
 $\pi_1((X, x_0) \xrightarrow{f} (Y, y_0)) := \mathbf{Aut}_{\Pi(X)}(x_0) \xrightarrow{\Pi(f)} \mathbf{Aut}_{\Pi(Y)}(y_0)$.

Proposition 2.1.3.

- If $[H] : f \xrightarrow{\sim} f' : X \rightarrow Y$ is a 2-cell in $2\text{-}\mathbf{Top}$ then $\pi_1(f')([\gamma]) = [H(x_0, -)] \circ \pi_1(f)([\gamma]) \circ [H(x_0, -)]^{-1}$, $[\gamma] \in \pi_1(X, x_0)$.
- In the case $[H] : f \xrightarrow{\sim} f' : (X, x_0) \rightarrow (Y, y_0)$ is a pointed 2-cell ($[H(x_0, -)] = 1_{f(x_0)} : f(x_0) \rightarrow f'(x_0) = f'(x_0)$) then $\pi_1(f) = \pi_1(f')$.

Proof follows from the naturality square

$$\begin{array}{ccc} f(x_0) & \xrightarrow{[H(x_0, -)] \sim} & f'(x_0) \\ \Pi(f)([\gamma]) \downarrow & & \downarrow \Pi(f')([\gamma]) \\ f(x_0) & \xrightarrow{[H(x_0, -)] \sim} & f'(x_0) \end{array} \quad \square$$

2.2. Duality and Invariant Theory.

Proposition 2.2.1. Let \mathbf{K} be \mathbf{Set} , \mathbf{Top} or \mathbf{Diff}^+ (spectra of smooth completion of commutative algebras with Zariski topology), G be a group. Then there exists a concrete natural

dual adjunction $\mathbf{ComAlg}^{op} \xrightleftharpoons[H]{F} G\text{-}\mathbf{K}$ with k (\mathbb{R} or \mathbb{C}), its schizophrenic object, such that

$k \in Ob\ G\text{-}\mathbf{K}$ has trivial action of G , and $F \circ H : G\text{-}\mathbf{K} \rightarrow G\text{-}\mathbf{K}$ is a functor 'taking the factor-space generated by equivalence relation $x \sim y$ iff $x, y \in \text{Closure}(\text{the same orbit})$ ' (it is essentially the orbit space). \square

Definition 2.2.1.

- Adjoint object $\mathcal{A}_X = HX$ for an object X in $G\text{-}\mathbf{K}$ is called its **algebra of invariants**.
- If $U : G\text{-}\mathbf{K} \rightarrow G\text{-}\mathbf{K}$ is an endofunctor then $\mathcal{A}_{U(X)}$ is called an **algebra of U -invariants** of object X . \square

Remarks.

- For $U = (-)^n$, n -fold Cartesian product, $\mathcal{A}_{U(X)}$ is an **n points' invariants' algebra**.
- For $\mathbf{K} = \mathbf{Diff}$, $U = \mathbf{Jet}^n$, $\mathbf{Jet}^n(X) := \{j_0^n f \mid f \in \mathbf{Diff}(k, X)\}$, set of all n -jets of all maps from k to X at point 0 (with a certain manifold structure obtained from local trivializations), we get **differential invariants**.
- $U = \mathbf{Jet}^\infty : \mathbf{Diff} \rightarrow \mathbf{Diff}^+$ does not fit to the above scheme, but everything is still correct if $U : G\text{-}\mathbf{K} \rightarrow G\text{-}\mathbf{K}_1$ is an extension to $G\text{-}\mathbf{K}_1$, a category concretely adjoint to \mathbf{ComAlg} .
- G can be, of course, $\mathbf{Aut}(X)$.

Accordingly to Klein's Erlangen Program every group acting on a space determines a geometry and, conversely, every geometry hides a group of transformations. Properties of geometric objects which do not change under all transformations are called geometric (or invariant, or absolute for the given G -space and a class of geometric objects).

Equivalence problem [Car1, Car2, Vas, Olv, Gar] consists of G -space X and two 'geometric objects' S_1, S_2 of the same type on space X . It is required to determine if these two objects can be mapped to one another by an element of G . An approach is to find a (complete) system of invariants of each object.

2.2.1. Classification of covariant geometric objects.

Under covariant geometric objects we mean objects like submanifold, foliation or system of differential equations, i.e., objects which behave contravariantly from Categorical viewpoint and which can be described by a **differential ideal** I ($dI \subset I$) in $\Lambda(X)$, exterior differential algebra of X .

Proposition 2.2.1.1. *Let G be a Lie-like group (i.e., there exists an algebra of invariant forms on G). Then any G -equivariant map $\sigma : G \rightarrow X$ (G is given with left shift action and X is a left G -space) produces a system of invariants of differential ideal $I \subset \Lambda(X)$ (with generators of degree 0 and 1) in the following way:*

- Take the image $\bar{\Lambda}_{inv} := \text{Im}(\Lambda_{inv}(G) \hookrightarrow \Lambda(G) \twoheadrightarrow \Lambda(G)/\sigma^*(I))$, where $\Lambda_{inv}(G)$ is a subalgebra of left-invariant forms on G , $\sigma^* : \Lambda(X) \rightarrow \Lambda(G)$ is the induced map of exterior differential algebras, $\sigma^*(I)$ is the smallest differential ideal in $\Lambda(G)$ containing image of I under σ^* .
- Take module $\Lambda^0(G) \cdot \bar{\Lambda}_{inv}^1$ generated by 1-forms in $\bar{\Lambda}_{inv}$ over $\Lambda^0(G)$. There is an open set $\mathcal{O} \subset G$ and a basis $\{\omega_{inv}^\alpha\}_{\alpha \in A} \subset \bar{\Lambda}_{inv}^1$ for module $\Lambda^0(G) \cdot \bar{\Lambda}_{inv}^1$ restricted on \mathcal{O} , i.e., $\forall \omega_{inv}^i \in \bar{\Lambda}_{inv}^1 \exists!$ functions $f_\alpha^i \in C^\infty(\mathcal{O})$ such that $\omega_{inv}^i = \sum_\alpha f_\alpha^i \omega_{inv}^\alpha$. Form set $J_0 := \{f_\alpha^i\}$.
- Take expansion of differentials $df_\alpha^i = \sum_\beta f_{\alpha\beta}^i \omega_{inv}^\beta$ (over \mathcal{O}). Form set $J_1 := \{f_{\alpha\beta}^i\}$.
- Continue this process to get $J_2 := \{f_{\alpha\beta\gamma}^i\}, \dots, J_n := \{f_{\alpha_1 \dots \alpha_{n+1}}^i\} \dots$. Form set $J := \bigcup_n J_n$. Its elements are relative invariants of differential ideal $I \subset \Lambda(X)$.
- Take algebra $\mathcal{A}_J \subset C^\infty(\mathcal{O})$, generated by J , and take its smooth completion $\overline{\mathcal{A}_J}$ (see 3.4). Then ideal $\mathbf{Rel}(\mathcal{A}_J) \twoheadrightarrow \overline{\mathbf{Alg}(J)} \twoheadrightarrow \overline{\mathcal{A}_J}$, of all relations of \mathcal{A}_J , gives absolute invariants of differential ideal $I \subset \Lambda(X)$, where $\overline{\mathbf{Alg}(J)}$ is the smooth completion of free algebra generated by J .

Proof follows from the diagrams

$$\begin{array}{ccc}
 G & \xrightarrow{l_g} & G \\
 \sigma \downarrow & & \downarrow \sigma \\
 X & \xrightarrow{l_g} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Lambda_{inv}(G) & \xleftarrow{id} & \Lambda_{inv}(G) \\
 \sigma^* \uparrow & & \uparrow \sigma^* \\
 \Lambda(X) & \xleftarrow{l_g^*} & \Lambda(X)
 \end{array}$$

and equations $\omega_{inv}^i = \sum_\alpha f_\alpha^i \omega_{inv}^\alpha \text{ mod } (\sigma^*(I))$. \square

Remark. $G\text{-Diff}(G, X)$ is in 1-1-correspondence with all sections of orbit space X_G . So, if X is homogenous then it is exactly the set of all points of X and $\sigma : G \rightarrow X = G \xrightarrow{\sim} G \times \{x_0\} \xrightarrow{1 \times i_{x_0}} G \times X \xrightarrow{\rho} X$ is a G -equivariant map corresponding to point $x_0 \in X$, where ρ is the given G -action on X .

Proposition 2.2.1.2 (Exterior differential algebra associated to a group of analytic automorphisms). *Let X be an analytic n -dimensional manifold, $\mathbf{An}(X)$, its group of automorphisms, $H^\infty(X) := \{j_0^\infty f \mid f \in \mathbf{Diff}(k^n, X), X \text{ is analytic, } \text{Jacobian}(f) \neq 0\}$, ∞ -frame bundle over X (with a usual topology and manifold structure). Then there is an exterior differential k -algebra $\Lambda_{inv}(H^\infty(X))$ of invariant forms on $H^\infty(X)$ freely generated by elements of degree 1 obtained by the following process:*

- $\omega^i := x_j^i dx^j$ are any 'shift' forms on X
- ω_j^i are most general solutions of Maurer-Cartan equations $d\omega^i = \omega_j^i \wedge \omega^j$
- ω_{jk}^i are most general solutions of Maurer-Cartan equations $d\omega_j^i = \omega_k^i \wedge \omega_j^k + \omega_{jk}^i \wedge \omega^k$
- $\omega_{jkl}^i, \dots, \omega_{i_1 \dots i_n}^i, \dots$

All forms are symmetric in lower indices. They characterize underlying space of $\mathbf{An}(X)$ uniquely up to analytic iso. \square

Remark. At each point $x_0 \in X$, $\omega^i = 0$, and forms $\bar{\omega}_{i_1 \dots i_n}^i := \omega_{i_1 \dots i_n}^i|_{\omega^i=0}$, $n \geq 1$, are free generators of exterior differential algebra of **differential group** acting simply transitively on each fiber of $H^\infty(X)$.

2.2.2. Classification of smooth embeddings into Lie group.

It is often the last step of smooth classification of geometric objects [Car2, Fin, Kob]. Process of finding of differential invariants is similar to that in Proposition 2.2.1.1.

Proposition 2.2.2.1. *For a smooth embedding $f : X \rightarrow G$ of smooth manifold X into Lie group G a complete system of differential invariants of f can be obtained in the following way:*

- $Im(f^* : \Lambda_{inv}^1(G) \rightarrow \Lambda(X))$ is locally free, so, take its basis $\omega_{inv}^i, i = 1, \dots, n$, $n = \dim(X)$, near each point.
- Coefficients of linear combinations $\omega_{inv}^I = \sum_{i=1}^n a_i^I \omega_{inv}^i$, $I = n+1, \dots, \dim(G)$, are differential invariants of first order (of map f).
- Coefficients of differentials of invariants of first order $da_i^I = \sum_{j=1}^n a_{ij}^I \omega_{inv}^j$ are differential invariants of second order (of map f).
- ... Coefficients of differentials of invariants of $(k-1)$ order $da_{i_1 \dots i_{k-1}}^I = \sum_{i_k=1}^n a_{i_1 \dots i_k}^I \omega_{inv}^{i_k}$ are differential invariants of order k ...

Such calculated invariants characterize orbit $G \cdot f$ uniquely up to 'changing parameter space' $X \xrightarrow{\sim} X'$.

Proof is straightforward. \square

3. Representable ∞ -functors

Definition 3.1. ∞ -categories L and L' are **equivalent** if $L \sim L'$ in ∞ -CAT. \square

If equivalence $L \sim L'$ is given by functors $L \begin{smallmatrix} \xrightarrow{F} \\ \sim \\ \xleftarrow{G} \end{smallmatrix} L'$ then $\forall a \in L^0$ $a \sim G \circ F(a)$, $\forall b \in L'^0$ $b \sim F \circ G(b)$ naturally in a and b .

Definition 3.2. ∞ -functor $F : L \rightarrow L'$ is (weakly)

- **faithful** if $\forall a, a' \in L^0$ $\forall f^n, g^n \in L^n(a, a')$ $F(f^n) \sim F(g^n) \Rightarrow f^n \sim g^n$,
- **full** if $\forall a, a' \in L^0$ $\forall h^n \in L'^n(F(a), F(a'))$ $\exists f^n \in L^n(a, a')$ such that $F(f^n) \sim h^n$,
- **surjective on objects** if $\forall b \in L'^0$ $\exists a \in L^0$ such that $F(a) \sim b$. \square

Unlike first order equivalence there is no simple criterion of higher order equivalence.

Proposition 3.1. *If functor $L \xrightarrow[\sim]{F} L'$ is an equivalence then F is (weakly) faithful full and surjective on objects.*

Proof. " \Rightarrow " Regard the diagram

$$\begin{array}{ccc}
 a & \xrightarrow[e^n \rho_a]{e^n \rho_a} & G \circ F(a) \\
 \downarrow f^n & \searrow e^n \theta_a & \downarrow G(F(f^n)) \\
 a' & \xrightarrow[e^n \theta_{a'}]{e^n \rho_{a'}} & G \circ F(a')
 \end{array}$$

where: $f^n \in L^n(a, a')$, $e^n \rho_a \in L^n(a, G(F(a)))$, $e^n \theta_a \in L^n(G(F(a)), a)$, $n \geq 0$.

Take $f^n, g^n : a \rightarrow a' \in L^n(a, a')$ such that $F(f^n) \sim F(g^n)$. Then $f^n \sim e^n \theta_{a'} \circ_{n+1} G(F(f^n)) \circ_{n+1} e^n \rho_a \sim e^n \theta_{a'} \circ_{n+1} G(F(g^n)) \circ_{n+1} e^n \rho_a \sim g^n$, i.e., F is faithful (G is faithful by symmetry).

Take $\alpha^n : F(a) \rightarrow F(a') \in L^n(F(a), F(a'))$. Then $\beta^n := e^n \theta_{a'} \circ_{n+1} G(\alpha^n) \circ_{n+1} e^n \rho_a : a \rightarrow a' \in L^n(a, a')$ is such that $G(F(\beta^n)) \sim G(\alpha^n)$. So, $F(\beta^n) \sim \alpha^n$ because G is faithful. Therefore, F is full (G is full by symmetry).

F and G are obviously surjective on objects. \square

Remark. The inverse direction " \Leftarrow " for the above proposition works only partially. Namely,

for each $b \in L^0$ choose $G(b) \in L^0$ and equivalence $b \begin{array}{c} \xrightarrow{\rho_b} \\ \sim \\ \xleftarrow{\theta_b} \end{array} F(G(b))$ (which is possible since F is surjective on objects), moreover, if $b = F(a)$ choose $G(b) = a$, $\rho_b = eb$, $\theta_b = e(F(G(b))) = eb$. For each $f^n : b \rightarrow b' \in L^n(b, b')$ choose an element $G(f^n) \in L^n(G(b), G(b'))$ such that $e^n \rho_{b'} \circ_{n+1} f^n \circ_{n+1} e^n \theta_b \sim F(G(f^n))$ (which is possible since F is fully faithful). Then $G : L' \rightarrow L$ is obviously a (weak) functor. $a = G(F(a))$ is natural in a by construction, but $b \sim F(G(b))$ is natural in b for only first order arrows ρ_b, θ_b presenting \sim . So, F should be somehow 'naturally surjective on objects' which does not make sense yet when functor G is not defined.

Definition 3.3. ∞ -functor $F : L \rightarrow L'$ is called

- **isomorphism** if it is a bijection (on sets L, L') and the inverse map is a functor,
- **quasiisomorphism** if there exists a functor $G : L' \rightarrow L$ such that $\forall a^n \in L^n \ G(F(a^n)) \sim a^n$ and $\forall b^n \in L'^n \ F(G(b^n)) \sim b^n$, $n \geq 0$. \square

Proposition 3.2. *Notions of (functor) isomorphism and quasiisomorphism coincide.*

Proof. Each isomorphism is a quasiisomorphism. Conversely, if $L \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} L'$ is a quasiisomorphism then $\forall a^n \in L^n$, $n \geq 0$, $G(F(ea^n)) \sim ea^n$. So, $d(G(F(ea^n))) = dea^n$, i.e. $G(F(dea^n)) = dea^n$ and $G(F(a^n)) = a^n$ (instead of d , c could be used). The same, $\forall b^n \in L'^n$, $n \geq 0$, $F(G(b^n)) = b^n$. \square

Denote (quasi)isomorphism (equivalence) relation by \simeq .

Examples (isomorphic ∞ -categories)

1. Assume, $f^n \xrightarrow[\alpha]{\simeq} g^n$ are isomorphic elements of degree n (in a strict category L) then $L(f^n, f^n) \simeq L(g^n, g^n)$ are isomorphic ∞ -categories. Indeed, there is an isomorphism $F :$

$L(f^n, f^n) \rightarrow L(g^n, g^n) : x \mapsto \alpha * (x * \alpha^{-1})$, where $*$ means a horizontal composite. F is a functor. Its inverse is $G : L(g^n, g^n) \rightarrow L(f^n, f^n) : y \mapsto \alpha^{-1} * (y * \alpha)$. [For α , just equivalence, it is not true]

2. $\infty\text{-CAT}(L(-, a), F) \simeq F(a)$ (see below Yoneda Lemma). \square

Definition 3.4. Two n -modifications $\alpha^n, \beta^n : L \rightarrow \infty\text{-CAT}$, $n \geq 0$, are called **quasiequivalent** of deepness k , $0 \leq k \leq n + 1$, (denote it by $\alpha^n \approx_k \beta^n$) if their corresponding components are quasiequivalent of deepness $k - 1$, i.e. $\forall a \in L^0 \alpha^n(a) \approx_{k-1} \beta^n(a)$. \approx_0 means \sim by definition. [In other words, $\alpha^n \approx_k \beta^n$ if all their components of components on deepness k are equivalent, i.e. $\alpha^n \approx_0 \beta^n$ if they are equivalent $\alpha^n \sim \beta^n$; $\alpha^n \approx_1 \beta^n$ if their components are equivalent $\forall a \in L^0 \alpha^n(a) \sim \beta^n(a)$; $\alpha^n \approx_2 \beta^n$ if components of all components are equivalent; etc.]. If $\alpha^n, \beta^n : L \rightarrow L'$ are proper n -modifications (living in $\infty\text{-CAT}$) for them only \approx_0 and \approx_1 make sense. \square

Lemma 3.1.

- \approx_k is an equivalence relation.
- $\approx_{k_1} \Rightarrow \approx_{k_2}$ if $k_1 \leq k_2$.
- If $\alpha^n \approx_k \beta^n$ then $d\alpha^n = d\beta^n$, $c\alpha^n = c\beta^n$.
- If $(L_1, \approx_{k_1}), (L_2, \approx_{k_2})$ are two ∞ -categories (not necessarily proper, i.e. living in $\infty\text{-CAT}$) for which given equivalence relations make sense for all elements, and $F : L_1 \rightarrow L_2, G : L_2 \rightarrow L_1$ are maps (not necessarily functors) such that $\forall l_1 \in L_1 G(F(l_1)) \approx_{k_1} l_1$ and $\forall l_2 \in L_2 F(G(l_2)) \approx_{k_2} l_2$, and F, G both preserve d (or c) then F, G are bijections inverse to each other.
- For $L, L' \in \text{Ob}(\infty\text{-CAT})$ and $a \in L^0$ the map $ev_a : \infty\text{-CAT}(L, L') \rightarrow L' : f^n \mapsto f^n(a)$ is a strict functor. [Similar statement holds when L, L' are not proper, e.g. $\infty\text{-CAT}$, but we need to formulate it a bigger universe containing $\infty\text{-CAT}$]

Proof. First two statements are obvious. Third one follows from the fact $x \sim y \Rightarrow dx = dy, cx = cy$ and that d, c are taken componentwise. Forth statement follows by the same argument as in the proof of proposition 1.3.2. The last statement holds because, again, all operations in $\infty\text{-CAT}(L, L')$ are taken componentwise. \square

Remark. For the proof of Yoneda lemma a double evaluation functor is needed. For two functors $F, G : L \rightarrow \infty\text{-CAT}$ take the restriction of evaluation functor ev_a on the hom-set between F and G , i.e. $ev_{a, F, G} : \infty\text{-CAT}(L, \infty\text{-CAT})(F, G) \rightarrow \infty\text{-CAT}(F(a), G(a)) : f^n \mapsto f^n(a)$, where $\infty\text{-CAT}$ is a bigger (and weaker) universe containing $\infty\text{-CAT}$ as an object. Now, take a second evaluation functor $ev_x : \infty\text{-CAT}(F(a), G(a)) \rightarrow G(a) : g^n \mapsto g^n(x), x \in (F(a))^0$. Then the double evaluation functor is the composite $ev_x \circ ev_{a, F, G} : \infty\text{-CAT}(L, \infty\text{-CAT})(F, G) \rightarrow G(a) : f^n \mapsto f^n(a)(x)$. It is a strict functor. \square

$\infty\text{-CAT}$ -valued functors, natural transformations and modifications live now in a bigger universe $\infty\text{-CAT}$, and we do not have for them appropriate definitions, yet.

Definition 3.5. $\infty\text{-CAT}$ -valued functors, natural transformations and modifications are introduced in a similar way as usual ones with changing all occurrences of \sim with (one degree weaker relation) \approx_1 , i.e.

- a map $F : L \rightarrow \infty\text{-CAT}$ of degree 0 is a **functor** if F strictly preserves d and c , $Fdx = dFx$, $Fcx = cFx$, and weakly up to \approx_1 preserves e and composites, $Fex \approx_1 eFx$, $F(x \circ_k y) \approx_1 F(x) \circ_k F(y)$,
- For a given sequence of two functors $F, G : L \rightarrow \infty\text{-CAT}$, ..., two $(n - 1)$ -modifications $\alpha_1^{n-1}, \alpha_2^{n-1} : \alpha_1^{n-2} \rightarrow \alpha_2^{n-2}$ strict (or weak) **n -modification** $\alpha^n : \alpha_1^{n-1} \rightarrow \alpha_2^{n-1}$ is a map

$\alpha^n : L^0 \rightarrow \infty\text{-}\mathbf{CAT}^{n+1}$ such that $\forall a, b \in L^0 \ \alpha^n(b) * F(-) \approx_1 G(-) * \alpha^n(a) : L^{\geq n}(a, b) \rightarrow L'^{\geq n}(F(a), G(b))$ (components of values of functors are equivalent). \square

Definition 3.6. Covariant (contravariant) functor $F : L \rightarrow \infty\text{-}\mathbf{CAT}$ is

- **weakly representable** if $\exists a \in L^0$ such that $L(a, -) \sim F(-)$ ($L(-, a) \sim F(-)$). It means there is an equivalence of two ∞ -categories $L(a, b) \sim F(b)$ ($L(b, a) \sim F(b)$) natural in b ,
- **strictly representable** if $\exists a \in L^0$ such that $L(a, -) \simeq F(-)$ ($L(-, a) \simeq F(-)$), i.e. $\forall b \in L^0 \ \exists$ isomorphism $L(a, b) \simeq F(b)$ ($L(b, a) \simeq F(b)$) natural in b . \square

Lemma 3.2. For given representable $L(-, a) : L^{op} \rightarrow \infty\text{-}\mathbf{CAT}$ and functor $F : L^{op} \rightarrow \infty\text{-}\mathbf{CAT}$

- all natural transformations $\tau^0 : L(-, a) \rightarrow F$ are of the form $\forall b \in Ob L$ b -component is a functor $\tau_b^0 : L(b, a) \rightarrow F(b)$, $\tau_b^0(f^m) \sim F(f^m)(\tau_a^0(ea))$, $f^m \in L^m(b, a)$,
- all n -modifications $\tau^n : L(-, a) \rightarrow F$, $n \geq 1$, are of the form $\forall b \in Ob L$ b -component is a $(n-1)$ -modification $\tau_b^n : L(b, a) \rightarrow F(b)$, $\tau_b^n(f^0) \sim F(f^0)(\tau_a^n(ea))$, $f^0 \in L^0(b, a)$.

Proof follows from the naturality square

$$\begin{array}{ccc} a & & L(a, a) \xrightarrow{\tau_a^n} F(a) \\ f^m \uparrow & L(f^m, a) \downarrow & \downarrow F(f^m) \\ b & & L(b, a) \xrightarrow{\tau_b^n} F(b) \end{array} \quad n \geq 0 \quad \square$$

Lemma 3.3. For a given n -cell $\beta^n \in (F(a))^n$, $n \geq 0$, n -modification $\tau^n : L(-, a) \rightarrow F$ such that $\tau_a^n(ea) = \beta^n$ exists and unique up to \approx_2 .

Proof. Uniqueness follows from lemma 3.2, existence from the definition of n -modification $\tau_b^n(f^m) := F(f^m)(\beta^n)$ (for $n > 0$, $m = 0$ only) and naturality square showing correctness of

the definition

$$\begin{array}{ccc} b & & L(b, a) \xrightarrow{\tau_b^n} F(b) \\ g^k \uparrow & L(g^k, a) \downarrow & \downarrow F(g^k) \\ c & & L(c, a) \xrightarrow{\tau_c^n} F(c) \end{array}$$

$(\mu_{c,b,a}(f^m, g^k) := \mu_{c,b,a}(e^{max(m,k)-m} f^m, e^{max(m,k)-k} g^k))$ \square

Corollary 1. All n -modifications $\tau^n : L(-, a) \rightarrow F$, $n \geq 0$, have strict form $\tau_b^n(f^0) = F(f^0)(\tau_a^n(ea))$, $f^0 \in L^0(b, a)$. \square

Corollary 2 (criterion of representability). $\infty\text{-}\mathbf{CAT}$ -valued presheaf $F : L^{op} \rightarrow \infty\text{-}\mathbf{CAT}$ is

- **strictly representable** (with representing object $a \in L^0$) iff there exists an object $\beta^0 \in (F(a))^0$ such that $\forall \gamma^n \in (F(b))^n$, $n \geq 0$, $\exists!$ n -arrow $(f^n : b \rightarrow a) \in L^n(b, a)$ with $\gamma^n = F(f^n)(\beta^0)$,
- **weakly representable** (with representing object $a \in L^0$) iff there exists an object $\beta^0 \in (F(a))^0$ such that $\forall b \in Ob L$ the functor $L(b, a) \rightarrow F(b) : f^n \mapsto F(f^n)(\beta^0)$ is an equivalence of categories.

(Similar statements hold for covariant presheaf $F : L \rightarrow \infty\text{-}\mathbf{CAT}$) \square

Proposition 3.3 (Yoneda Lemma). For functor $F : L^{op} \rightarrow \infty\text{-}\mathbf{CAT}$ and object $a \in L^0$ there is a strict isomorphism $\infty\text{-}\mathbf{CAT}(L(-, a), F) \simeq F(a)$ natural in a and F .

Proof. Strict functoriality of the correspondence $\tau^n \mapsto \tau_a^n(ea)$ is straightforward (because it is a double evaluation functor). The map $\beta^n \mapsto F(-)(\beta^n)$ is quasiinverse to the first map (with

respect to \approx_2 and $=$ equivalence relations in $\infty\text{-CAT}(L(-, a), F)$ and $F(a)$ respectively), and it strictly preserves d and c . So, these both maps are strict isomorphisms.

$$\begin{array}{ccc} & \begin{array}{c} a \\ \uparrow f^m \\ b \end{array} & \begin{array}{c} F \\ \downarrow \alpha^k \\ G \end{array} \\ \text{Naturality is given by} & & \begin{array}{ccc} \infty\text{-CAT}(L(-, a), F) & \xrightarrow{\simeq} & F(a) \\ \downarrow \infty\text{-CAT}(L(-, f^m), \alpha^k) & & \downarrow \alpha^k(f^m) \\ \infty\text{-CAT}(L(-, b), G) & \xrightarrow{\simeq} & G(b) \end{array} \end{array}$$

(where $\alpha^k(f^m) := \mu_{F(a), F(b), G(b)}(e^{\max(k, m)-k} \alpha_b^k, e^{\max(k, m)-m+1} F(f^m))$, $k, m \geq 0$) \square

Remark. Yoneda lemma for ∞ -categories is similar to one for first order categories with the difference that elements $\beta^n \in (F(a))^n$ of degree n determine now higher degree arrows (n -modifications) $\beta^n : L(-, a) \rightarrow F$ in $\infty\text{-CAT}$ -valued presheaves category. \square

Proposition 3.4 (Yoneda embedding). *There is Yoneda embedding $\mathbf{Y} : L \rightarrow \infty\text{-CAT}^{L^{op}}$: $\alpha \mapsto L(-, \alpha)$, $\alpha \in L$, which is an extension of the isomorphisms from Yoneda lemma determined on hom-sets $L(a, b)$, $a, b \in L^0$. Yoneda embedding preserves and reflects equivalences \sim .*

Proof. By Yoneda isomorphism for a given $f^n \in L^n(a, b)$ the corresponding n -modification is $L(-, b)(f^n) : L(-, a) \rightarrow L(-, b)$ which is the same as $L(f^n, -) : L(-, a) \rightarrow L(-, b)$, i.e. functor $\mathbf{Y} : L \rightarrow \infty\text{-CAT}^{L^{op}}$: $\alpha \mapsto L(-, \alpha)$, $\alpha \in L$, locally coincides with isomorphisms from Yoneda lemma. By lemma 1.3 this functor preserves and reflects equivalences \sim . \square

Remark. Under assumption that a category $\infty\text{-CAT}$ of **weak** categories, functors and n -modifications exist all the above reasons remain essentially the same, i.e. Yoneda lemma and embedding seem to hold in a weak situation. \square

4. (Co)limits

Definition 4.1. ∞ -**graph** is a graded set $G = \coprod_{n \geq 0} G^n$ with two unary operations $d, c : \coprod_{n \geq 1} G^n \rightarrow \coprod_{n \geq 0} G^n$ of degree -1 such that $d^2 = dc$, $c^2 = cd$. \square

Definition 4.2. ∞ -**diagram** $D : G \rightarrow L$ from ∞ -graph G to ∞ -category L is a function of degree 0 which preserves operations d, c . \square

Proposition 4.1. *All diagrams from G to L , natural transformations, modifications form ∞ -category $\mathbf{Dgrm}_{G, L}$ in the same way as functor category $\infty\text{-CAT}(L', L)$. \square*

For a given object $a \in L^0$ **constant diagram** to a is $\Delta(a) : G \rightarrow L : g \mapsto e^n a$ if $g \in G^n$. $\Delta : L \rightarrow \mathbf{Dgrm}_{G, L}$ is an ∞ -functor.

Denote $\{e\}\alpha := \{\alpha, e\alpha, e^2\alpha, \dots, e^n\alpha, \dots\}$, $\alpha \in L$.

Definition 4.3. Diagram $D : G \rightarrow L$ has

- **limit** if functor $\mathbf{Dgrm}_{G, L}(\Delta(-), D) : L^{op} \rightarrow \infty\text{-CAT}$ is representable.

If $\nu : L(-, a) \xrightarrow{\sim} \mathbf{Dgrm}_{G, L}(\Delta(-), D)$ is the equivalence then

$\nu_a(\{e\}ea) \subset \mathbf{Dgrm}_{G, L}(\Delta(a), D)$ is called **limit cone** over D , a is its **vertice** (or diagram **limit** $\lim D$), $\nu_a(ea)$ are its **edges**, $\nu_a(e^k a)$, $k > 1$, are identities up to \sim

- **colimit** if functor $\mathbf{Dgrm}_{G, L}(D, \Delta(-)) : L \rightarrow \infty\text{-CAT}$ is representable.

If $\nu : L(a, -) \xrightarrow{\sim} \mathbf{Dgrm}_{G, L}(D, \Delta(-))$ is the equivalence then

$\nu_a(\{e\}ea) \subset \mathbf{Dgrm}_{G, L}(D, \Delta(a))$ is called **colimit cocone** over D , a is its **vertice** (or diagram **colimit** $\text{colim } D$), $\nu_a(ea)$ are its **edges**, $\nu_a(e^k a)$, $k > 1$, are identities up to \sim \square

Remark. Conditions on equivalence ν in the above definition can be strengthened. If it is a (natural) isomorphism then (co)limits are called **strict** and as a rule they are different from **weak** ones [Bor1].

Proposition 4.2. *For strict (co)limits the following is true*

$$\bullet \quad L \begin{array}{c} \xleftarrow{\quad \text{lim} \quad} \\ \xrightarrow{\quad \text{colim} \quad} \end{array} \mathbf{Dgrm}_{G,L}$$

$\begin{array}{c} \top \\ \Delta \\ \top \end{array}$

- *Strict right adjoints preserve limits (strict left adjoints preserve colimits).*

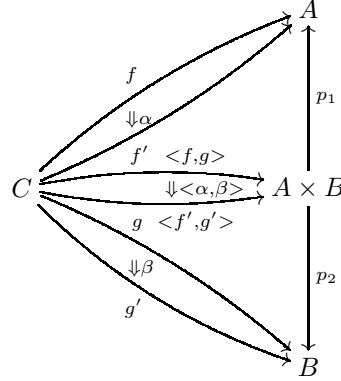
Proof.

- It is immediate from definition 4.3 and proposition 5.1.
- The argument is the same as for first order categories (see example 13.a, point 1) [the essential thing is that a strict adjunction is determined by (triangle) identities which are preserved under ∞ -functors]. \square

Examples

1. (strict binary products in **2-Top** and **2-CAT**) They coincide with '1-dimensional' products.

Mediating 2-cell arrow is given componentwise



2. ('equalizer' of a 2-cell in **2-CAT**) [Bor1] For a given 2-cell $\mathbf{A} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} \mathbf{B}$ in **2-CAT** its strict

limit is a subcategory $\mathbf{E} \hookrightarrow \mathbf{A}$ such that $F(A) = G(A)$ and $\alpha_A = 1_{F(A)} : F(A) \rightarrow G(A)$ (on objects), and $F(f) = G(f)$ (on arrows).

3. (strict and weak pullbacks in **2-CAT**) [Bor1] Let \mathcal{P} be a '2-dimensional' graph $1 \xrightarrow{x} 0 \xleftarrow{y} 2$ with trivial 2-cells, $F : \mathcal{P} \rightarrow \mathbf{2-CAT}$ be a 2-functor. Then its limit is a pullback diagram in **2-CAT**

$$\begin{array}{ccccc} F(1) \times_{F(0)} F(2) & \xrightarrow{p_2} & F(2) & & \\ p_1 \downarrow & \searrow p_3 & \downarrow F(y) & & \\ F(1) & \xrightarrow{F(x)} & F(0) & & \end{array} \quad \text{When the limit is taken **strictly** } F(1) \times_{F(0)} F(2) \text{ coincides}$$

with '1-dimensional' pullback, i.e. $F(1) \times_{F(0)} F(2) \hookrightarrow F(1) \times F(2)$ is a subcategory consisting of objects (A, B) , $A \in \text{Ob } F(1)$, $B \in \text{Ob } F(2)$, $F(x)(A) = F(y)(B)$ and arrows (f, g) , $f \in \text{Ar } F(1)$, $g \in \text{Ar } F(2)$, $F(x)(f) = F(y)(g)$. When the limit is taken **weakly** $F(1) \times_{F(0)} F(2)$ is not a subcategory of product $F(1) \times F(2)$. It consists of 5-tuples (A, B, C, f, g) , $A \in \text{Ob } F(1)$,

$B \in \text{Ob } F(2)$, $C \in \text{Ob } F(0)$, $f : F(x)(A) \xrightarrow{\sim} C$, $g : F(y)(B) \xrightarrow{\sim} C$ are isomorphisms, with arrows (a, b, c) , $a : A \rightarrow A'$, $b : B \rightarrow B'$, $c : C \rightarrow C'$ such that $c \circ f = f' \circ F(x)(a)$, $c \circ g = g' \circ F(y)(b)$. Projections p_1, p_2, p_3 are obvious. The pullback square commutes up to isomorphisms $f : F(x) \circ p_1 \Rightarrow p_3$, $g : F(y) \circ p_2 \Rightarrow p_3$. \square

5. Adjunction

Definition 5.1. The situation $L \begin{smallmatrix} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{smallmatrix} L'$ (where L, L' are ∞ -categories, F, G are ∞ -functors) is called

- **weak ∞ -adjunction** if there is an equivalence $L(-, G(+)) \sim L'(F(-), +) : L^{op} \times L' \rightarrow \infty\text{-CAT}$ (i.e. $L(a, G(b)) \sim L'(F(a), b)$ natural in $a \in L^0$, $b \in L'^0$),
- **strict ∞ -adjunction** if there is an isomorphism $L(-, G(+)) \simeq L'(F(-), +) : L^{op} \times L' \rightarrow \infty\text{-CAT}$ (i.e. $L(a, G(b)) \simeq L'(F(a), b)$ natural in $a \in L^0$, $b \in L'^0$). \square

Proposition 5.1. *The following are equivalent*

1. $L \begin{smallmatrix} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{smallmatrix} L'$ is a strict ∞ -adjunction
2. $\forall b \in L'^0$ $L'(F(-), b)$ is strictly representable
3. $\forall a \in L^0$ $L(a, G(-))$ is strictly representable

Proof.

- 1. \implies 2., 3. is immediate
- 2. \implies 1. From criterion of strict representability (see point 1.3) it follows that $\forall b \in L'^0$ there exists a 'universal element' $(\beta_b^0 : F(G(b)) \rightarrow b) \in L'^0(F(G(b)), b)$ such that $\forall (f^n : F(c) \rightarrow b) \in L'^n(F(c), b) \exists ! n\text{-arrow } (g^n : c \rightarrow G(b)) \in L^n(c, G(b))$ with $f^n = \mu_{F(c), F(G(b)), b}(e^n \beta_b^0, F(g^n))$

$$\begin{array}{ccc} G(b) & & F(G(b)) \xrightarrow{e^n \beta_b^0} b \\ \uparrow \exists ! g^n & \uparrow F(g^n) & \nearrow \forall f^n \\ c & F(c) & \end{array}$$

Consequently, $\forall (f^n : b' \rightarrow b) \in L'^n(b', b)$ the diagram holds

$$\begin{array}{ccccc} G(b) & & F(G(b)) & \xrightarrow{e^n \beta_b^0} & b \\ \uparrow G(f^n) & & \uparrow F(G(f^n)) & & \uparrow f^n \\ G(b') & & F(G(b')) & \xrightarrow{e^n \beta_{b'}^0} & b' \end{array}$$

It shows that assignment $\text{Ob } L' \ni b \mapsto G(b) \in \text{Ob } L$ is extendable to a functor $G : L' \rightarrow L$ (in an essentially unique way) and that $\beta^0 : FG \rightarrow 1_{L'}$ is a natural transformation (**counit** ε of the adjunction $F \dashv G$).

Isomorphism $\varphi_{c,b} : L'(F(c), b) \rightarrow L(c, G(b))$ such that

$$\begin{array}{ccc} F(G(b)) & \xrightarrow{e^n \beta_b^0} & b \\ \uparrow F(\varphi_{c,b}(f^n)) & \nearrow f^n & \\ F(c) & & \end{array} \quad \text{is natural in}$$

$c \in \text{Ob } L$, $b \in \text{Ob } L'$ because of the naturality square

$$\begin{array}{ccc}
\begin{array}{c} c \\ \uparrow g^n \\ c' \end{array} & \begin{array}{c} b \\ \downarrow f^n \\ b' \end{array} & \begin{array}{ccc} L'(F(c), b) & \xrightarrow{\varphi_{c,b}} & L(c, G(b)) \\ \downarrow L'(F(g^n), f^n) & & \downarrow L(g^n, G(f^n)) \\ L'(F(c'), b') & \xrightarrow{\varphi_{c',b'}} & L(c', G(b')) \end{array}
\end{array}$$

(indeed, $\forall h^n \in L'^n(F(c), b)$ $G(f^n) * \varphi_{c,b}(h^n) * g^n \sim \varphi_{c',b'}(f^n * h^n * F(g^n))$, where $*$ is the horizontal composite, since $e^n \beta_b^0 * F(G(f^n) * \varphi_{c,b}(h^n) * g^n) \sim f^n * e^n \beta_b^0 * F(\varphi_{c,b}(h^n)) * F(g^n) \sim f^n * h^n * F(g^n)$)

- 3. \implies 1. is similar to 2. \implies 1. □

Remark. Analogous statement for a weak ∞ -adjunction is not true. In the above proof 'universal elements' were essentially used. □

Definition 5.2. For a given **strict** ∞ -adjunction $L \overset{F}{\underset{G}{\rightleftarrows}} L'$

- universal elements $\varepsilon_b : F(G(b)) \rightarrow b$ representing functors $L'(F(-), b)$ ($b \in Ob L'$ is a parameter) form natural transformation $\varepsilon : FG \rightarrow 1_{L'}$ which is called **counit** of the adjunction,
- universal elements $\eta_a : a \rightarrow G(F(a))$ representing functors $L(a, G(-))$ ($a \in Ob L$ is a parameter) form natural transformation $\eta : 1_L \rightarrow GF$ which is called **unit** of the adjunction. □

Remark. For a **weak** ∞ -adjunction no usefull unit and counit exist. □

Proposition 5.2.

- For both weak and strict adjunctions: composite of left adjoints is a left adjoint (composite of right adjoints is a right adjoint).
- For a weak (strict) adjunction a right or left adjoint is determined uniquely up to equivalence \sim (up to isomorphism \simeq).

Proof.

- If $L \overset{F}{\underset{G}{\rightleftarrows}} L' \overset{F'}{\underset{G'}{\rightleftarrows}} L''$ then $L''(F'Fl, l'') \sim L'(Fl, G'l'') \sim L(l, GG'l'')$ (composite of natural equivalences). [For a strict adjunction the same reason works]
- Assume, $L'(a, G'b) \sim L(Fa, b) \sim L'(a, Gb)$ are natural equivalences then $L'(-, G'b) \sim L'(-, Gb)$ is a natural transformation (equivalence) natural in b . Then, by Yoneda embedding, $G'b \sim Gb$ naturally in b , i.e. $G' \sim G$. [Again, changing \sim with \simeq still works]. □

Proposition 5.3. For a strict adjunction $L \overset{F}{\underset{G}{\rightleftarrows}} L'$ Kan definition and definition via 'unit-counit' coincide, i.e. the following are equivalent

- $\varphi_{a,b} : L(a, G(b)) \simeq L'(F(a), b) : \varphi_{a,b}^*$ natural in $a \in L^0, b \in L'^0$,
- \exists natural transformations $\eta : 1_L \rightarrow GF$ and $\varepsilon : FG \rightarrow 1_{L'}$ satisfying triangle identities $\varepsilon F \circ_1 F \eta = 1_F$ and $G \varepsilon \circ_1 \eta G = 1_G$.

Proof. For a strict adjunction the same proof as for first order categories works.

- Universal elements η_a, ε_b for functors $L(a, G(-)), L'(F(-), b)$ mean that they are images of $1_{F(a)}, 1_{G(b)}$ under functors $\varphi_{a,F(a)}^*, \varphi_{G(b),b}$, i.e.

$$\begin{array}{ccc}
FGFa & \xrightarrow{\varepsilon_{Fa}} & Fa \\
\uparrow F\eta_a & \nearrow 1_{Fa} & \\
Fa & & \\
\downarrow 1_{Fa} & &
\end{array}
\quad
\begin{array}{ccc}
Gb & \xrightarrow{\eta_{Gb}} & GFGB \\
\downarrow 1_{Gb} & \searrow & \downarrow G\varepsilon_b \\
Gb & & Gb
\end{array}$$

(strict equalities)

- Define maps $\begin{cases} \varphi_{a,b}(f^n) := e^n(\varepsilon_b) \circ_{n+1} F(f^n), & f^n \in L^n(a, G(b)) \\ \varphi_{a,b}^*(g^n) := G(g^n) \circ_{n+1} e^n(\eta_a), & g^n \in L'^n(F(a), b) \end{cases}$

They are functors $\begin{cases} \varphi_{a,b} := \varepsilon_b * F(-) : L(a, G(b)) \rightarrow L'(F(a), b) \\ \varphi_{a,b}^* := G(-) * \eta_a : L'(F(a), b) \rightarrow L(a, G(b)) \end{cases}$ and inverses to each other:

$$\begin{aligned} \varphi_{a,b}^*(\varphi_{a,b}(f^n)) &= \varphi_{a,b}^*(e^n \varepsilon_b \circ_{n+1} F(f^n)) = G(e^n \varepsilon_b \circ_{n+1} F(f^n)) \circ_{n+1} e^n \eta_a = e^n G(\varepsilon_b) \circ_{n+1} \\ (GF(f^n) \circ_{n+1} e^n \eta_a) &= e^n G(\varepsilon_b) \circ_{n+1} (e^n \eta_{G(b)} \circ_{n+1} f^n) = e^{n+1} G(b) \circ_{n+1} f^n = f^n, \text{ and similar,} \\ \varphi_{a,b}(\varphi_{a,b}^*(g^n)) &= g^n. \end{aligned}$$

$$\begin{array}{ccc} L(a, G(b)) & \xrightarrow{\varphi_{a,b}} & L'(F(a), b) \\ \downarrow L(x^m, G(y^m)) & & \downarrow L'(F(x^m), y^m) \\ L(a', G(b')) & \xrightarrow{\varphi_{a',b'}} & L'(F(a'), b') \end{array}$$

Naturality (e.g., of $\varphi_{a,b}$) follows from the square. $(\varphi_{a',b'}(L(x^m, G(y^m))(f^n))) = \varphi_{a',b'}(G(y^m) * f^n * x^m) = \varepsilon_{b'} * FG(y^m) * F(f^n) * F(x^m) = y^m * \varepsilon_b * F(f^n) * F(x^m) = L'(F(x^m), y^m)(\varphi_{a,b}(f^n))$, where $n = 0$ or $m = 0$. \square

Examples of higher order adjunction

1. Every usual 1-adjunction $A \overset{\perp}{\rightleftarrows} B$ is ∞ -1-adjunction for trivial ∞ -extensions of A and B .
2. Gelfand-Naimark dual 1-adjunction $\mathbf{C}^* \mathbf{Alg}^{op} \overset{\perp}{\rightleftarrows} \mathbf{CHTop}$ is extendable to ∞ -2-adjunction (see point 9).
3. Quillen theorem [Mac]. Let Δ be a category of finite linearly ordered sets, $\mathbf{Set}^{\Delta^{op}}$ category of simplicial sets, $Ho(\mathbf{Top}) := (2\text{-}\mathbf{Top})^{(1)}$, $Ho(\mathbf{Set}^{\Delta^{op}}) := (2\text{-}\mathbf{Set}^{\Delta^{op}})^{(1)}$. Then

$$\begin{array}{ccc} \Delta & & \\ \downarrow \text{Yoneda} & \searrow & \\ \mathbf{Set}^{\Delta^{op}} & \overset{\perp}{\rightleftarrows} & \mathbf{Top} \\ \downarrow & & \downarrow \\ Ho(\mathbf{Set}^{\Delta^{op}}) & \overset{\perp}{\rightleftarrows} & Ho(\mathbf{Top}) \end{array}$$

 \square

So, the top adjunction is actually 2-adjunction (or ∞ -2-adjunction). All the above adjunctions are strict.

6. Concrete duality for ∞ -categories

Duality preserves all categorical properties. It is the one extremity of functors (in the sense of invariants) especially useful when the duality is concrete. It is significant that concrete duality for ∞ -categories behaves the same as for 1-categories.

Definition 6.1.

- **Duality** is just an equivalence $L^{op} \sim L'$.

- **Concrete duality** over $\mathbb{B} \hookrightarrow \infty\text{-CAT}$ is a duality $L^{op} \begin{smallmatrix} \xrightarrow{G} \\ \sim \\ \xleftarrow{F} \end{smallmatrix} L'$ such that \exists (faithful) forgetful functors $U : L \rightarrow \mathbb{B}$, $V : L' \rightarrow \mathbb{B}$ and objects $\tilde{A} \in L^0$, $\tilde{B} \in L'^0$ such that
 - $U(\tilde{A}) \sim V(\tilde{B})$,

- $V \circ_1 G \sim L(-, \tilde{A})$, $U \circ_1 F^{op} \sim L'(-, \tilde{B})$

$$\begin{array}{ccc} L^{op} & \xrightarrow{G} & L' \\ & \searrow & \downarrow V \\ & L(-, \tilde{A}) & \downarrow \\ & & \mathbb{B} \end{array} \quad \begin{array}{ccc} L'^{op} & \xrightarrow{F^{op}} & L \\ & \searrow & \downarrow U \\ & L'(-, \tilde{B}) & \downarrow \\ & & \mathbb{B} \end{array}$$

Representing objects $\tilde{A} \in L^0$, $\tilde{B} \in L'^0$ are called together a **dualizing** or **schizophrenic object** for the given concrete duality[P-Th].

[for a **concrete dual adjunction** the definition is similar] \square

Proposition 6.1 (representable forgetfuls \Rightarrow concrete dual adjunction). *Let (L, U) ,*

(L', V) be (weakly) dually adjoint ∞ -categories $L^{op} \begin{smallmatrix} \xrightarrow{G} \\ \top \\ \xleftarrow{F} \end{smallmatrix} L'$ with representable forgetful functors $U \sim L(A_0, -) : L \rightarrow \mathbb{B}$, $V \sim L'(B_0, -) : L' \rightarrow \mathbb{B}$ (where $\mathbb{B} \hookrightarrow \infty\text{-CAT}$ is a subcategory). Then this adjunction is concrete over \mathbb{B} with dualizing object (\tilde{A}, \tilde{B}) , where $\tilde{A} := F(B_0)$, $\tilde{B} := G(A_0)$, i.e.

- $U(\tilde{A}) \sim V(\tilde{B})$

- $V \circ_1 G \sim L(-, \tilde{A})$, $U \circ_1 F^{op} \sim L'(-, \tilde{B})$

$$\begin{array}{ccc} L^{op} & \xrightarrow{G} & L' \\ & \searrow & \downarrow V \\ & L(-, \tilde{A}) & \downarrow \\ & & \mathbb{B} \end{array} \quad \begin{array}{ccc} L'^{op} & \xrightarrow{F^{op}} & L \\ & \searrow & \downarrow U \\ & L'(-, \tilde{B}) & \downarrow \\ & & \mathbb{B} \end{array}$$

Proof.

- $U(\tilde{A}) = UF(B_0) \sim L(A_0, FB_0) \sim L'(B_0, GA_0) \sim VGA_0 = V\tilde{B}$
- $VG(-) \sim L'(B_0, G(-)) \sim L(-, FB_0) = L(-, \tilde{A})$ (and similar, $UF(-) \sim L'(-, \tilde{B})$) \square

Remarks.

- Concrete duality as above should be called **weak**. **Strict** variants of definition 6.1 and proposition 6.1 also exist (by changing \sim with isomorphism \simeq and weak dual adjunction with the strict one).
- (Weak or strict) concrete duality (dual adjunction) is given essentially by hom-functors which admit lifting along forgetful functors (to obtain proper values). Representing objects of these functors have equivalent (or isomorphic) underlying objects.
- For usual 1-dimensional categories $\mathbb{B} = \mathbf{Set} \hookrightarrow \infty\text{-CAT}$ (∞ -1-subcategory). For dimension n , as a rule, $\mathbb{B} = n\text{-Cat} \hookrightarrow \infty\text{-CAT}$ (∞ - n -subcategory of small $(n-1)$ -categories). \square

6.1. Natural and non natural duality.

Definition 6.1.1.

- For hom-set $L(A, \tilde{A})$ and element $(x : A_0 \rightarrow A) \in L^0(A_0, A)$ **evaluation functor** at point x is $ev_{A,x} := L(x, \tilde{A}) : L(A, \tilde{A}) \rightarrow L(A_0, \tilde{A})$ ($ev_{A,x} \in \mathbb{B}^1 \hookrightarrow \infty\text{-CAT}^1$). Similarly, **evaluation** $(n-1)$ -**modification** ev_{A,x^n} , $n = 1, 2, \dots$, for $x^n \in L^n(A_0, A)$ is $L(x^n, \tilde{A}) \in \mathbb{B}^n(L(A, \tilde{A}), L(A_0, \tilde{A}))$.

- For a forgetfull functor $V : L' \rightarrow \mathbb{B}$ an arrow $f^n : V(Y) \rightarrow V(Y') \in \mathbb{B}^n(V(Y), V(Y'))$ is called **L' -arrow** if $\exists \Phi^n : Y \rightarrow Y' \in L'^n(Y, Y')$ such that $V(\Phi^n) = f^n$.
- Lifting of hom-functor $V \circ G \sim L(-, \tilde{A})$ is called **initial** [A-H-S] if $\forall A \in L^0 \forall Y \in L'^0 \forall f^n : V(Y) \rightarrow L(A, \tilde{A}) \in \mathbb{B}^n(V(Y), L(A, \tilde{A}))$ f^n is an L' -arrow iff $\forall (x^n : A_0 \rightarrow A) \in L^n(A_0, A)$ $ev_{A, x^n} \circ_{n+1} f^n : V(Y) \rightarrow L(A_0, \tilde{A}) \in \mathbb{B}^n(V(Y), L(A_0, \tilde{A}))$ is an L' -arrow.
- If liftings of hom-functors $V \circ G \sim L(-, \tilde{A})$, $U \circ F \sim L'(-, \tilde{B})$ are both initial then the concrete dual adjunction $L^{op} \begin{smallmatrix} \xrightarrow{G} \\ \top \\ \xleftarrow{F} \end{smallmatrix} L'$, if exists, is called **natural** [Hof, P-Th], and otherwise, non natural.

Even if $U\tilde{A} \sim V\tilde{B}$ and $\forall A \in L^0, B \in L'^0$ \mathbb{B} -objects $L(A, \tilde{A})$, $L'(B, \tilde{B})$ can be lifted to L', L hom-functors $L(-, \tilde{A})$, $L'(-, \tilde{B})$ need not to be lifted (which happens only if lifting of the assignments $A \mapsto L(A, \tilde{A})$, $B \mapsto L'(B, \tilde{B})$ can be extended functorially over all cells).

Initial lifting condition for evaluation cones

$\{ev_{A, x^n} \in \mathbb{B}^n(L(A, \tilde{A}), L(A_0, \tilde{A}))\}_{x^n \in L^n(A_0, A)}^{n \in \mathbb{N}}$, $\{ev_{B, y^n} \in \mathbb{B}^n(L'(B, \tilde{B}), L'(B_0, \tilde{B}))\}_{y^n \in L'^n(B_0, B)}^{n \in \mathbb{N}}$ consists of the following requirements:

- hom-categories of the form $L(A, \tilde{A})$, $L'(B, \tilde{B}) \in Ob(\mathbb{B})$ can be lifted to L', L
- evaluation cones $\{ev_{A, x^n} \in \mathbb{B}^n(L(A, \tilde{A}), L(A_0, \tilde{A}))\}_{x^n \in L^n(A_0, A)}^{n \in \mathbb{N}}$, $\{ev_{B, y^n} \in \mathbb{B}^n(L'(B, \tilde{B}), L'(B_0, \tilde{B}))\}_{y^n \in L'^n(B_0, B)}^{n \in \mathbb{N}}$ can be lifted to $\{ev_{A, x^n} \in L'^n(G(A), \tilde{B})\}_{x^n \in L^n(A_0, A)}^{n \in \mathbb{N}}$, $\{ev_{B, y^n} \in L^n(F(B), \tilde{A})\}_{y^n \in L'^n(B_0, B)}^{n \in \mathbb{N}}$ in L', L
- $\forall f^n \in \mathbb{B}^n(VX, L(A, \tilde{A}))$ f^n is L' -arrow iff $\forall x^n \in L^n(A_0, A)$ $\mu(ev_{A, x^n}, f^n) \in \mathbb{B}^n(VX, L(A_0, \tilde{A}))$ is L' -arrow (and, symmetrically, $\forall g^n \in \mathbb{B}^n(UY, L'(B, \tilde{B}))$ g^n is L -arrow iff $\forall y^n \in L'^n(B_0, B)$ $\mu(ev_{B, y^n}, g^n) \in \mathbb{B}^n(UY, L'(B_0, \tilde{B}))$ is L -arrow) \square

Denote further (in the the following proof) lifted evaluation maps by $ev_{A, x}$ (or like that) and underlying evaluation maps in \mathbb{B} by $|ev_{A, x}|$.

Proposition 6.1.1. *If two strict ∞ -categories L, L' concrete over $\mathbb{B} \hookrightarrow \infty\text{-CAT}$ with representable (strictly faithful) forgetful functors $U = L(A_0, -)$, $V = L'(B_0, -)$ have objects $\tilde{A} \in L^0$, $\tilde{B} \in L'^0$ such that*

- $U\tilde{A} \sim V\tilde{B}$
- hom-functors $L(-, \tilde{A}) : L^{op} \rightarrow \mathbb{B}$, $L'(-, \tilde{B}) : L'^{op} \rightarrow \mathbb{B}$ satisfy **initial lifting condition for evaluation cones**

then \exists natural strict concrete dual adjunction $L^{op} \begin{smallmatrix} \xrightarrow{G} \\ \top \\ \xleftarrow{F} \end{smallmatrix} L'$ $L(A, FB) \underset{\text{nat. iso}}{\simeq} L'(B, GA)$

$$\begin{array}{ccc} L^{op} & \xrightarrow{G} & L' \\ & \searrow & \downarrow V \\ & L(-, \tilde{A}) & \downarrow \\ & & \mathbb{B} \end{array} \quad \begin{array}{ccc} L'^{op} & \xrightarrow{F^{op}} & L \\ & \searrow & \downarrow U \\ & L'(-, \tilde{B}) & \downarrow \\ & & \mathbb{B} \end{array} \quad \text{with } (\tilde{A}, \tilde{B}), \text{ its schizophrenic object.}$$

Proof.

- $L(A, \tilde{A})$, $L'(B, \tilde{B})$ are lifted to L', L by condition.
- Let $f^n \in L^n(A, A')$, then $L(f^n, \tilde{A}) : L(A', \tilde{A}) \rightarrow L(A, \tilde{A})$ is L' -arrow since $ev_{A, a^n} \circ_{n+1} L(f^n, \tilde{A}) := L(a^n, \tilde{A}) \circ_{n+1} L(f^n, \tilde{A}) = L(f^n \circ_{n+1} a^n, \tilde{A}) =: ev_{A', f^n \circ_{n+1} a^n}$, which is liftable $\forall a^n \in L^n(A_0, \tilde{A})$. Therefore, $L(f^n, \tilde{A})$ is L' -arrow, and similarly, $L'(g^n, \tilde{B})$ is L -arrow, i.e., \exists

maps $L^{op} \begin{array}{c} \xrightarrow{G} \\ \xleftarrow{F} \end{array} L'$, which are obviously functorial. Why do they give an adjunction?

- (unit and counit) 1-arrow (unit) $\eta_B : B \rightarrow GFB$ is given by $|\eta_B| =: V\eta_B : |B| \rightarrow |GFB| : b \mapsto [ev_{B,b} : FB \rightarrow \tilde{A}], b \in |B| = L'(B_0, B), |GFB| = L(FB, \tilde{A}), |ev_{B,b}| : |FB| \rightarrow |\tilde{A}|, |FB| = L'(B, \tilde{B}), |\tilde{A}| = L(A_0, \tilde{A}) \sim L'(B_0, \tilde{B})$. Why can $|\eta_B|$ be lifted to L' ? Take composite with evaluation maps $|ev_{FB,c}| \circ_1 |\eta_B|(b) = |ev_{FB,c}|(ev_{B,b}) = |ev_{B,b}|(c) = |c|(b)$, where $c \in |FB|^0 = L'^0(B, \tilde{B}) = L^0(A_0, FB)$, $b \in |B|^n$. So, $|ev_{FB,c}| \circ_1 |\eta_B| = |c|$ is L' -arrow. Therefore, $|\eta_B|$ is L' -arrow. Counit is given symmetrically $\varepsilon_A \rightarrow FGA$, $|\varepsilon_A| : |A| \rightarrow |FGA| : a \mapsto [ev_{A,a} : GA \rightarrow \tilde{B}], |A| = L(A_0, A), |FGA| = L'(GA, \tilde{B}), |ev_{A,a}| : |GA| \rightarrow |\tilde{B}|, |GA| = L(A, \tilde{A}), |\tilde{B}| = L'(B_0, \tilde{B}) \sim L(A_0, \tilde{A})$. By the same argument $|\varepsilon_A|$ is an L -arrow.
- (triangle identities) $G\varepsilon_A \circ_1 \eta_{GA} = 1_{GA}$, $F\eta_B \circ_1 \varepsilon_{FB} = 1_{FB}$. It is sufficient to prove them for underlying maps. Since forgetful functors are faithful they will follow.

$|G\varepsilon_A| \circ_1 |\eta_{GA}| \stackrel{?}{=} |1_{GA}|$, where $|\eta_{GA}| : |GA| \rightarrow |GFGA|$, $|GA| = L(A, \tilde{A})$, $|GFGA| = L(FGA, \tilde{A})$, $\varepsilon_A : A \rightarrow FGA$, $|G\varepsilon_A| : |GFGA| \rightarrow |GA|$.

Take $(f^n : A \rightarrow \tilde{A}) \in |GA| = L^n(A, \tilde{A})$, $a^m \in |A| = L^m(A_0, A)$. Two cases are possible

$$\begin{cases} (a) (f^n, n > 0) \ \& \ (a^0) : ||G\varepsilon_A| \circ_1 |\eta_{GA}|(f^n)|(a^0) = |L(\varepsilon_A, \tilde{A})(ev_{GA, f^n})|(a^0) = |ev_{GA, f^n} \circ_{n+1}| \\ (b) (f^0) \ \& \ (a^n, n \geq 0) : ||G\varepsilon_A| \circ_1 |\eta_{GA}|(f^0)|(a^n) = |L(\varepsilon_A, \tilde{A})(ev_{GA, f^0})|(a^n) = |ev_{GA, f^0} \circ_1| \\ (a) e^n \varepsilon_A|(a^0) = |ev_{GA, f^n}| \circ_{n+1} e^n |\varepsilon_A|(a^0) = |ev_{GA, f^n}|(ev_{A, e^n a^0}) = |ev_{A, e^n a^0}|(f^n) = |f^n|(a^0) \\ (b) \varepsilon_A|(a^n) = |ev_{GA, f^0}| \circ_1 |\varepsilon_A|(a^n) = |ev_{GA, f^0}|(ev_{A, a^n}) = |ev_{A, a^n}|(f^0) = |f^0|(a^n) \\ (a) =: \mu_{A_0, A, \tilde{A}}^L(f^n, e^n a^0) = ||1_{GA}|(f^n)|(a^0) \\ (b) =: \mu_{A_0, A, \tilde{A}}^L(e^n f^0, a^n) = ||1_{GA}|(f^0)|(a^n) \end{cases}$$

Second triangle identity holds similarly.

- (naturality of η_B, ε_A) Again, it is sufficient to prove naturality for underlying maps

$$\begin{array}{ccc} |B| & \xrightarrow{|\eta_B|} & |GFB| \\ |f| \downarrow & & \downarrow |GFf| = L(Ff, \tilde{A}) \\ |B'| & \xrightarrow{|\eta_{B'}|} & |GFB'| \end{array} \quad \text{Two cases are } \begin{cases} (a) (b^n \in |B|^n, n \geq 0) \ \& \ (f^0 \in L'^0(B, B')) \\ (b) (b^0 \in |B|^0) \ \& \ (f^n \in L'^n(B, B')) \end{cases}$$

$$(a) \quad \begin{array}{ccc} |B| & \xrightarrow{|\eta_B|} & |GFB| \\ |f^0| \downarrow & & \downarrow |GFf^0| = L(Ff^0, \tilde{A}) \\ |B'| & \xrightarrow{|\eta_{B'}|} & |GFB'| \end{array}$$

$$(b) \quad \begin{array}{ccc} |B| & \xrightarrow{e^n |\eta_B|} & |GFB| \\ |f^n| \downarrow & & \downarrow |GFf^n| = L(Ff^n, \tilde{A}) \\ |B'| & \xrightarrow{e^n |\eta_{B'}|} & |GFB'| \end{array}$$

$$\begin{array}{ccc} b^n & \xrightarrow{\quad} & ev_{B, b^n} \\ \downarrow & & \downarrow \\ ev_{B, b^n} \circ_{n+1} e^n (Ff^0) & & \\ \parallel & & \\ \downarrow & & \\ |f^0|(b^n) & \xrightarrow{\quad} & ev_{B', |f^0|(b^n)} \\ b^0 & \xrightarrow{\quad} & ev_{B, e^n b^0} \\ \downarrow & & \downarrow \\ ev_{B, e^n b^0} \circ_{n+1} (Ff^n) & & \\ \parallel & & \\ \downarrow & & \\ |f^n|(b^0) & \xrightarrow{\quad} & ev_{B', |f^n|(b^0)} \end{array}$$

$$(\text{recall } |f^n|(b^0) \equiv \mu(f^n, e^n b^0), \quad |f^0|(b^n) \equiv \mu(e^n f^0, b^n))$$

$$\text{Why } \begin{cases} (a) & ev_{B,b^n} \circ_{n+1} e^n(Ff^0) = ev_{B',|f^0|(b^n)} \\ (b) & ev_{B,e^n b^0} \circ_{n+1} (Ff^n) = ev_{B',|f^n|(b^0)} \end{cases} ?$$

Take underlying maps:

$$\begin{cases} (a) & |ev_{B,b^n}| \circ_{n+1} e^n |Ff^0|(h^n) = |ev_{B,b^n}|(h^n \circ_{n+1} e^n f^0) = |h^n \circ_{n+1} e^n f^0|(b^n) = \\ (b) & |ev_{B,e^n b^0}| \circ_{n+1} |Ff^n|(h^0) = |ev_{B,e^n b^0}|(e^n h^0 \circ_{n+1} f^n) = |e^n h^0 \circ_{n+1} f^n|(e^n b^0) = \\ (a) & = |h^n| \circ_{n+1} |e^n f^0|(b^n) = |ev_{B',|f^0|(b^n)}|(h^n), \quad h^n \in L'^n(B', \tilde{B}) \\ (b) & = |e^n h^0| \circ_{n+1} |f^n|(e^n b^0) = |ev_{B',|f^n|(b^0)}|(h^0), \quad h^0 \in L'^0(B', \tilde{B}) \end{cases}$$

(types of the above arrows are $Ff : FB' \rightarrow FB$, $ev_{B,b} : FB \rightarrow \tilde{A}$ (L -map), $ev_{B',|f|(b)} FB' \rightarrow \tilde{A}$ (L -map), $|ev_{B,b}| : L'(B, \tilde{B}) \rightarrow |\tilde{B}| = L'(B_0, \tilde{B})$, $|ev_{B',|f|(b)}| : L'(B', \tilde{B}) \rightarrow |\tilde{B}| = L'(B_0, \tilde{B})$, $|Ff| : L'(B', \tilde{B}) \rightarrow L'(B, \tilde{B})$, $|Ff| = L'(f, \tilde{B})$).

Therefore, η_B is natural. Similarly, ε_A is natural.

$$\bullet \text{ (iso-functors } L(A, FB) \xrightleftharpoons[\theta_{A,B}^*]{\theta_{A,B}} L'(B, GA) \text{)}$$

$$\text{Define } \begin{cases} \theta_{A,B}(f^n) := G(f^n) \circ_{n+1} e^n(\eta_B), & f^n \in L^n(A, FB) \\ \theta_{A,B}^*(g^n) := F(g^n) \circ_{n+1} e^n(\varepsilon_A), & g^n \in L'^n(B, GA) \end{cases}$$

Let $g^n \in L'^n(B, GA)$. Then $\theta_{A,B}(\theta_{A,B}^*(g^n)) := G(Fg^n \circ_{n+1} e^n(\varepsilon_A)) \circ_{n+1} e^n(\eta_B) = e^n(G\varepsilon_A) \circ_{n+1} GFg^n \circ_{n+1} e^n(\eta_B) \stackrel{\text{nat. of } \eta_B}{=} e^n(G\varepsilon_A) \circ_{n+1} e^n(\eta_{GA}) \circ_{n+1} g^n \stackrel{\text{triangle id.}}{=} e^n(1_{GA}) \circ_{n+1} g^n = e^{n+1}(GA) \circ_{n+1} g^n = g^n$. Similarly, $\theta_{A,B}^*(\theta_{A,B}(f^n)) = f^n$, $f^n \in L^n(A, FB)$. $\theta_{A,B}$, $\theta_{A,B}^*$ are obviously functors. So, they are isomorphisms.

- (naturality of $\theta_{A,B}$, $\theta_{A,B}^*$) We need to prove diagram

$$\begin{array}{ccccc} A & B & L(A, FB) & \xrightarrow{e^n \theta_{A,B}} & L'(B, GA) \\ x^n \uparrow & y^n \uparrow & \downarrow L(x^n, Fy^n) & & \downarrow L'(y^n, Gx^n) \\ A' & B' & L(A', FB') & \xrightarrow{e^n \theta_{A',B'}} & L'(B', GA') \end{array}$$

$$L'(y^n, Gx^n) \circ_{n+1} e^n \theta_{A,B} \stackrel{?}{=} e^n \theta_{A',B'} \circ_{n+1} L(x^n, Fy^n)$$

Two cases are: $\begin{cases} (a) & (f^0 \in L(A, FB)) \ \& \ (x^n, y^n, n > 0) \\ (b) & (f^n \in L(A, FB), n \geq 0) \ \& \ (x^0, y^0) \end{cases}$

$$(a) \quad \begin{array}{ccc} f^0 & \xrightarrow{\quad} & e^n G(f^0) \circ_{n+1} e^n(\eta_B) \\ \downarrow & & \downarrow \\ & & Gx^n \circ_{n+1} (e^n G(f^0) \circ_{n+1} e^n(\eta_B)) \circ_{n+1} y^n \\ & & \parallel \\ & & = \parallel ? \\ & & \downarrow \\ Fy^n \circ_{n+1} e^n f^0 \circ_{n+1} x^n & \xrightarrow{\quad} & G(Fy^n \circ_{n+1} e^n f^0 \circ_{n+1} x^n) \circ_{n+1} e^n(\eta_{B'}) = (\eta_B \text{ is nat.}) \\ & & \parallel \\ & & = \parallel \\ & & \downarrow \\ & & Gx^n \circ_{n+1} e^n Gf^0 \circ_{n+1} GFy^n \circ_{n+1} e^n(\eta_{B'}) \end{array}$$

$$\begin{array}{ccc}
f^n & \xrightarrow{\quad} & G(f^n) \circ_{n+1} e^n(\eta_B) \\
\downarrow & & \downarrow \\
& & e^n Gx^0 \circ_{n+1} (G(f^n) \circ_{n+1} e^n(\eta_B)) \circ_{n+1} e^n y^0 \\
& & \parallel \\
& & = \parallel ? \\
& & \downarrow \\
e^n Fy^0 \circ_{n+1} f^n \circ_{n+1} e^n x^0 & \xrightarrow{\quad} & G(e^n Fy^0 \circ_{n+1} f^n \circ_{n+1} e^n x^0) \circ_{n+1} e^n(\eta_{B'}) = (\eta_B \text{ is nat.}) \\
& & \parallel \\
& & = \parallel \\
& & \downarrow \\
& & e^n Gx^0 \circ_{n+1} Gf^n \circ_{n+1} e^n GFy^0 \circ_{n+1} e^n(\eta_{B'})
\end{array}$$

Therefore, L and L' are concretely dually adjoint. This correspondence is natural (by condition) and strict ($\theta_{A,B}$ and $\theta_{A,B}^*$ are isomorphisms). \square

Corollary. Concrete natural duality is a **strict** adjunction. \square

Well-known dualities [P-Th, Bel, A-H-S]

All dualities below are of first order, natural [P-Th], and obtained by restriction of appropriate dual adjunctions.

1. \mathbf{Vec}_k is dually equivalent to itself $\mathbf{Vec}_k^{op} \xrightleftharpoons[\mathbf{Vec}_k(-,k)]{\mathbf{Vec}_k(-,k)} \mathbf{Vec}_k$, where \mathbf{Vec}_k is a category of vector spaces over field k

2. $\mathbf{Set}^{op} \sim \mathbf{Complete Atomic Boolean Algebras}$

3. $\mathbf{Bool}^{op} \sim \mathbf{Boolean Spaces}$ (Stone duality), where \mathbf{Bool} is a category of Boolean rings (every element is idempotent). It is obtained from the dual adjunction $\mathbf{CRing} \xrightleftharpoons[\mathbf{Top}(-,2)]{\mathbf{CRing}(-,2)} \mathbf{Top}$,

where $\mathbf{2}$ is two-element ring and discrete topological space. $\mathbf{CRing}(A, \mathbf{2}) \hookrightarrow \mathbf{2}^A$ (subspace in Tychonoff topology)

4. $\text{hom}(-, \mathbb{R}/\mathbb{Z}) : \mathbf{CompAb}^{op} \sim \mathbf{Ab}$ (Pontryagin duality), where \mathbf{CompAb} , \mathbf{Ab} are categories of compact abelian groups and abelian groups respectively
5. $\text{hom}(-, \mathbb{C}) : \mathbf{C*Alg}^{op} \sim \mathbf{CHTop}$ (Gelfand-Naimark duality), where $\mathbf{C*Alg}$, \mathbf{CHTop} are categories of commutative \mathbb{C}^* -algebras and compact Hausdorff spaces. $\mathbf{C*Alg}(A, \mathbb{C}) \hookrightarrow \mathbb{C}^A$ (subspace in Tychonoff topology)

7. Vinogradov duality

Let K be a commutative ring, A a commutative algebra over K , $A\text{-Mod} \hookrightarrow K\text{-Mod}$ be categories of modules over A and K respectively.

Definition 7.1. [V-K-L] For $P, Q \in \text{Ob}(A\text{-Mod})$

- K -linear maps $l(a) := a \cdot -, r(a) := - \cdot a, \delta(a) := l(a) - r(a) : K\text{-Mod}(P, Q) \rightarrow K\text{-Mod}(P, Q)$ are called **left, right multiplications** and **difference operator** (by element $a \in A$),
- K -linear map $\Delta : P \rightarrow Q$ is a **differential operator of order $\leq r$** if $\forall a_0, a_1, \dots, a_r \in A$ $\delta_{a_0, a_1, \dots, a_r}(\Delta) = 0$, where $\delta_{a_0, a_1, \dots, a_r} := \delta_{a_0} \circ \delta_{a_1} \circ \dots \circ \delta_{a_r}$. \square

Lemma 7.1.

- If $\Delta_1 \in K\text{-Mod}(P, Q)$, $\Delta_2 \in K\text{-Mod}(Q, R)$ are differential operators of order $\leq r$ and $\leq s$ respectively, then $\Delta_2 \circ \Delta_1 : K\text{-Mod}(P, R)$ is a differential operator of order $\leq r + s$,
- $\forall a \in A$, $P \in \text{Ob}(A\text{-Mod})$ module multiplication (by a) $l_a : P \rightarrow P : p \mapsto ap$ is a differential operator of order 0. \square

All differential operators between A -modules form a category $A\text{-Diff}$, such that $A\text{-Mod} \hookrightarrow A\text{-Diff} \hookrightarrow K\text{-Mod}$, and first two categories have the same objects. $A\text{-Diff}$ is enriched in $(K\text{-Mod}, \otimes_K)$ in a proper sense and enriched in two different ways in $(A\text{-Mod}, \otimes_K)$ loosing composition property to be A -module map. Module multiplication for the first enrichment $A\text{-Diff}$ in $(A\text{-Mod}, \otimes_K)$ is given by $A \times A\text{-Diff}(P, Q) \rightarrow A\text{-Diff}(P, Q) : (a, \Delta) \mapsto l_a \circ \Delta$, for the second enrichment by $A \times A\text{-Diff}(P, Q) \rightarrow A\text{-Diff}(P, Q) : (a, \Delta) \mapsto \Delta \circ l_a$. Denote $A\text{-Diff}$ with left module multiplication in hom-sets $l_a \circ -$ by the same name $A\text{-Diff}$ and with right multiplication in hom-sets $- \circ l_a$ by $A\text{-Diff}^+$.

Proposition 7.1.

- $\forall P, Q \in \text{Ob}(A\text{-Mod})$ $A\text{-Diff}(P, Q) = \bigcup_{s=0}^{\infty} \text{Diff}_s(P, Q)$, $A\text{-Diff}^+(P, Q) = \bigcup_{s=0}^{\infty} \text{Diff}_s^+(P, Q)$ are filtered A -modules by submodules of differential operators of order $\leq s$, $s = 0, 1, \dots$,
- $\forall P \in \text{Ob}(A\text{-Mod})$ $A\text{-Diff}(P, P)$ is an associative K -algebra. \square

Proposition 7.2.

- $\text{Diff}_s(P, -)$, $\text{Diff}_s^+(-, P) : A\text{-Mod} \rightarrow A\text{-Mod}$ are A -linear functors,
- $\forall P \in \text{Ob}(A\text{-Mod})$ functor $\text{Diff}_s^+(-, P)$ is representable by object $\text{Diff}_s^+(A, P) := \text{Diff}_s^+(A, P)$, i.e. $\forall Q \in \text{Ob}(A\text{-Mod})$ $A\text{-Mod}(Q, \text{Diff}_s^+(A, P)) \xrightarrow{\sim} \text{Diff}_s^+(Q, P)$,
- $\forall P \in \text{Ob}(A\text{-Mod})$ functor $\text{Diff}_s(P, -)$ is representable by object $\text{Jet}^s(P) := A \otimes_K P \text{ mod } \mu^{s+1}$, where μ^{s+1} is a submodule of $A \otimes_K P$ generated by elements $\delta^{a_0} \circ \dots \circ \delta^{a_{s+1}}(a \otimes p)$ [$\delta^b(a \otimes p) := ab \otimes p - a \otimes bp$], i.e. $\forall Q \in \text{Ob}(A\text{-Mod})$ $A\text{-Mod}(\text{Jet}^s(P), Q) \xrightarrow{\sim} \text{Diff}^s(P, Q)$,
- inclusion $A\text{-Mod} \hookrightarrow A\text{-Diff}^+$ is an (enriched) left adjoint with counit $ev : \text{Diff}^+(P) \rightarrow P : \Delta \mapsto \Delta(1)$, i.e. $\forall \Delta \in \text{Diff}^+(Q, P) \exists ! f_\Delta \in A\text{-Mod}(Q, \text{Diff}^+(P))$ such that

$$\begin{array}{ccc} \text{Diff}^+(P) & \xrightarrow{ev} & P \\ \uparrow f_\Delta & \nearrow \Delta & \\ Q & & \end{array}$$

and this correspondence is A -linear, $f_\Delta : q \mapsto (a \mapsto \Delta(aq))$,

- inclusion $A\text{-Mod} \hookrightarrow A\text{-Diff}$ is an (enriched) right adjoint with unit $j^\infty : P \rightarrow \text{Jet}^\infty(P) : p \mapsto 1 \otimes p \text{ mod } \mu^\infty$ [$\mu^\infty := \bigcap_{s=0}^{\infty} \mu^s$], i.e. $\forall \Delta \in \text{Diff}(P, Q) \exists ! f^\Delta \in A\text{-Mod}(\text{Jet}^\infty(P), Q)$ such that

$$\begin{array}{ccc} P & \xrightarrow{j^\infty} & \text{Jet}^\infty(P) \\ & \searrow \Delta & \downarrow f^\Delta \\ & & Q \end{array}$$

and this correspondence is A -linear, $f^\Delta : (a \otimes p) \text{ mod } \mu^\infty \mapsto a\Delta(p)$,

- subcategory $A\text{-Mod}$ is reflective and coreflective in $A\text{-Diff}$ (enriched in $K\text{-Mod}$). \square

$\forall s \in \mathbb{N}$ introduce two full subcategories of $A\text{-Mod}$:

- **A-Mod-Diff_s**, consisting of all A -modules of type $\mathbf{Diff}_s(P, A)$, $P \in \mathbf{Ob}(A\text{-Mod})$, and A -module A ,
- **A-Mod-Jet^s**, consisting of all A -modules of type $\mathbf{Jet}^s(P)$, $P \in \mathbf{Ob}(A\text{-Mod})$, and A -module A .

Proposition 7.3 (Vinogradov Duality).

There is a duality $A\text{-Mod-Diff}_s^{op} \xrightleftharpoons{\sim} A\text{-Mod-Jet}^s$, $s \in \mathbb{N}$, concrete over $A\text{-Mod}$, namely,

$\mathbf{Diff}_s(P, A) \simeq A\text{-Mod}(\mathbf{Jet}^s(P), A)$, $\mathbf{Jet}^s(P) \simeq A\text{-Mod}(\mathbf{Diff}^s(P, A), A)$. A is a schizophrenic object. \square

Remarks.

- The above proposition states a formal analogue of duality between differential operators and jets over a fixed manifold X . Geometric modules of sections of vector bundles over X correspond to modules P over $C^\infty(X)$ with property $\bigcap_{x \in X} \mu_x P = 0$, where μ_x is a maximal ideal at point $x \in X$. Functors $\mathbf{Diff}_s(-, A)$ and $\mathbf{Jet}^s(-)$ preserve module property to be geometric [V-K-L].
- This duality is an alternative (algebraic) way to introduce jet-bundles in Geometry (instead of classical approach due to Grothendieck and Ehresmann as equivalence classes of maps tangent of order s at a point). When $A = C^\infty(X)$ and P is a geometric module realizable as a vector bundle $V(P)$ over X then $\mathbf{Jet}^s(P)$ is realizable as $\mathbf{Jet}^s(V(P))$ over X in classical sense [V-K-L, Vin1, Vin2]. \square

8. Duality for differential equations

Proposition 8.1. Let \mathbf{UAlg} be a category of universal algebras with representable forgetful functor. Then every topological algebra \mathfrak{A} is a schizophrenic object (see [P-Th]), and so, yields a natural dual adjunction between \mathbf{UAlg} and \mathbf{Top} .

Proof.

- Initial topology on $\mathbf{UAlg}(B, \mathfrak{A})$ gives initial lifting with respect to evaluation maps $ev_{B,b} : \mathbf{UAlg}(B, \mathfrak{A}) \rightarrow |\mathfrak{A}|$, $b \in |B|$.
- Algebra of continuous functions $\mathbf{Top}(X, \mathfrak{A})$ is initial with respect to evaluation maps $ev_{X,x} : \mathbf{Top}(X, \mathfrak{A}) \rightarrow |\mathfrak{A}|$, $x \in |X|$ (which are obviously homomorphisms) since operations in $\mathbf{Top}(X, \mathfrak{A})$ are pointwise and each arrow $f \in \mathbf{Top}(X, \mathfrak{A})$ is completely determined by all its values $ev_{X,x}(f) = |f|(x)$, $x \in |X|$. So that, if $g : |B| \rightarrow \mathbf{Top}(X, \mathfrak{A})$ is a **Set**-map such that $\forall x \in |X|$ $ev_{X,x} \circ g$ is a homomorphism $(\omega_n(ev_{X,x} \circ g)b_1, \dots, (ev_{X,x} \circ g)b_n = ev_{X,x} \circ g \omega_n b_1, \dots, b_n = ev_{X,x} \omega_n g b_1, \dots, g b_n)$, where ω_n is an n -ary operation. First equality holds because $ev_{X,x} \circ g$ is a homomorphism, second equality because $ev_{X,x}$ is a homomorphism), then g is a homomorphism since two maps whose values coincide at each point coincide themselves. \square

Corollary. Take $\mathbf{UAlg} = k\text{-}\Lambda\text{-Alg}$ (category of exterior differential algebras over a field k (\mathbb{R} or \mathbb{C}) which presents differential equations). Take $\mathfrak{A} = \Lambda(C^\infty(\mathbb{R}^n))$ or $\Lambda(C^\omega(\mathbb{C}^n))$ (which presents a parameter space) with a topology not weaker than jet^∞ . Then there exists a natural dual adjunction $k\text{-}\Lambda\text{-Alg}^{op} \xrightleftharpoons{\perp} \mathbf{Top}$ (between differential equations and their solution spaces). \square

Remark. If we regard full category $k\text{-}\Lambda\text{-Alg}$ whose forgetful functor is representable we will get a lot of extra 'points' which do not have geometric sense. Only graded maps of degree 0 to \mathfrak{A} have geometric sense (they present integral manifolds of dimension not bigger than n). In this

case representation of exterior differential algebras when it exists will be not via their solution spaces but via much bigger spaces. If we restrict $k\text{-}\Lambda\text{-}\mathbf{Alg}$ to only graded morphisms of degree 0 then forgetful functor is not representable. But the notion of 'schizophrenic object' still makes sense and theorem for natural dual adjunction [P-Th] still holds. So, there is a representation of exterior differential algebras via their usual solution spaces. \square

Denote concrete subcategories of **Top** dual to categories $k\text{-}\mathbf{Alg}$ (algebras over k) and $k\text{-}\Lambda\text{-}\mathbf{Alg}$ (exterior differential algebras over k with graded degree 0 morphisms) by **alg-Sol** and **diff-Sol** respectively, i.e., $k\text{-}\mathbf{Alg}^{op} \sim \mathbf{alg-Sol}$, $k\text{-}\Lambda\text{-}\mathbf{Alg}^{op} \sim \mathbf{diff-Sol}$. In particular, **alg-Sol** contains all algebraic and all smooth k -manifolds ($k = \mathbb{R}$ or \mathbb{C}), **diff-Sol** contains all spaces like $\mathbf{alg-Sol}(k^n, X)$ (with representing object $\mathfrak{A} = \Lambda(C^\infty(k^n))$).

Lemma 8.1 (rough structure of **diff-Sol**).

- $Ob(\mathbf{diff-Sol})$ are pairs $(X, \coprod_{i=1}^n \mathcal{F}_i)$ where $X := k\text{-}\Lambda\text{-}\mathbf{Alg}(D, k) = k\text{-}\mathbf{Alg}(D, k) \in Ob(\mathbf{alg-Sol})$, $\mathcal{F}_i \subset \mathbf{alg-Sol}(k^i, X)$, $1 \leq i \leq n$ [\mathcal{F}_i are not arbitrary subspaces of $\mathbf{alg-Sol}(k^i, X)$].
- $Ar(\mathbf{diff-Sol})$ are pairs $(f, \coprod_{i=1}^n \mathbf{alg-Sol}(k^i, f)) : (X, \coprod_{i=1}^n \mathcal{F}_i) \rightarrow (X', \coprod_{i=1}^n \mathcal{F}'_i)$ where $f : X \rightarrow X' \in Ar(\mathbf{alg-Sol})$, $\mathbf{alg-Sol}(k^i, f) : \mathcal{F}_i \rightarrow \mathcal{F}'_i$, $1 \leq i \leq n$. \square

Proposition 8.2. *There are following adjunctions*

- $k\text{-}\mathbf{Alg} \begin{array}{c} \xrightarrow{\Lambda_k} \\ \perp \\ \xleftarrow{p_0} \end{array} k\text{-}\Lambda\text{-}\mathbf{Alg}$ where Λ_k is the **free exterior differential algebra functor** (see 4.2.1), p_0 is the projection onto subalgebra of 0-elements,
 - $\mathbf{alg-Sol} \begin{array}{c} \xrightarrow{\text{hom}(k^n, -)} \\ \top \\ \xleftarrow{b} \end{array} \mathbf{diff-Sol}$ where b is taking the base space,
- such that

$$\begin{array}{ccc}
 k\text{-}\Lambda\text{-}\mathbf{Alg}^{op} & \begin{array}{c} \xrightarrow{F} \\ \sim \\ \xleftarrow{F'} \end{array} & \mathbf{diff-Sol} \\
 \Lambda_k^{op} \uparrow \vdash p_0^{op} & \text{hom}(k^n, -) & \downarrow \vdash b \\
 k\text{-}\mathbf{Alg}^{op} & \begin{array}{c} \xrightarrow{G} \\ \sim \\ \xleftarrow{G'} \end{array} & \mathbf{alg-Sol}
 \end{array}$$

Proof.

- $k\text{-}\Lambda\text{-}\mathbf{Alg}(\Lambda_k(A), D) \xrightarrow{\sim} k\text{-}\mathbf{Alg}(A, p_0(D))$ (natural in A and D)

$$\begin{array}{ccc}
 \in \uparrow & & \uparrow \in \\
 \rho & \xrightarrow{\sim} & \rho_0
 \end{array}$$

where ρ_0 is the 0-component of graded degree 0 homomorphism $\rho = \bigoplus_{i \geq 0} \rho_i$.

- $\mathbf{diff-Sol}(S, \text{hom}(k^n, X)) \xrightarrow{\sim} \mathbf{alg-Sol}(b(S), X)$ (natural in S and X)

$$\begin{array}{ccc}
 \in \uparrow & & \uparrow \in \\
 f & \xrightarrow{\sim} & f
 \end{array}$$

where: S is a pair $(b(S), \coprod_{i=1}^n \mathcal{F}_i)$, $\mathcal{F}_i \subset \text{hom}(k^i, b(S))$, $1 \leq i \leq n$, right $f : b(S) \rightarrow X$ is a usual map, and left $f := (f, \coprod_{i=1}^n \text{hom}(k^i, f)) : (b(S), \coprod_{i=1}^n \mathcal{F}_i) \rightarrow (X, \coprod_{i=1}^n \text{hom}(k^i, X))$.

The above square of adjunctions is immediate. \square

8.1. Cartan involution.

For systems in Cartan involution a (single) solution can be calculated recursively beginning from smallest 0 dimension. By Cartan's theorem [BC3G, Car1, Fin, Vas] every system can be made into such a form by sufficient number of differential prolongations [BC3G, Car1, Fin, Vas]. There is a cohomological criterion for systems to be in the involution.

Definition 8.1.1. Let $\mathcal{A} \in \text{Ob}(k\text{-}\Lambda\text{-}\mathbf{Alg})$, \mathfrak{A}_n be $\Lambda_{\mathbb{R}}(C^\infty(\mathbb{R}^n))$ or $\Lambda_{\mathbb{C}}(C^\omega(\mathbb{C}^n))$, $n \geq 0$.

- Any (differential homomorphism of degree 0) $\rho : \mathcal{A} \rightarrow \mathfrak{A}_n$ is called an **integral manifold** of \mathcal{A} (of dimension not bigger than n).
- $\deg(\rho : \mathcal{A} \rightarrow \mathfrak{A}_n) = m$, $0 \leq m \leq n$, iff ρ can be factored through a $\gamma : \mathcal{A} \rightarrow \mathfrak{A}_m$, i.e.,

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\gamma} & \mathfrak{A}_m \\ & \searrow \rho & \downarrow \\ & & \mathfrak{A}_n \end{array} \quad \text{and } m \text{ is the smallest such number.}$$

- $\deg(\mathcal{A}) = n$ iff maximal degree of integral manifolds of \mathcal{A} is n .
- \mathcal{A} , $\deg(\mathcal{A}) = n$, is in **Cartan involution** iff for each m -dimensional integral manifold $\rho : \mathcal{A} \rightarrow \mathfrak{A}_m$, $m < n$, there exists an $(m+1)$ -dimensional integral manifold $\beta : \mathcal{A} \rightarrow \mathfrak{A}_{m+1}$ which

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\exists \beta} & \mathfrak{A}_{m+1} \\ & \searrow \forall \rho & \downarrow \\ & & \mathfrak{A}_m \end{array}$$

contains ρ , i.e., \square

Remarks.

- \mathfrak{A}_0 is just k (\mathbb{R} or \mathbb{C}) with trivial differential. $\rho : \mathcal{A} \rightarrow \mathfrak{A}_0$ corresponds to a point $b(\rho) : b(\mathcal{A}) \rightarrow k$. Each point of \mathcal{A} is a 0-dimensional integral manifold.
- Original Cartan's definition was for classical algebras (factor-algebras of $\Lambda_{\mathbb{R}}(C^\omega(\mathbb{R}^N))$) and in terms of 'infinitesimal integral elements' (nondifferential homomorphisms of degree 0 $f : \mathcal{A} \rightarrow \Lambda_k(d\tau^1, \dots, d\tau^N)$) [BC3G, Car1, Fin]. For that case both definitions coincide.
- By a number of differential prolongations (adding new jet-variables with obvious relations) every classical system can be made into Cartan involution form (E. Cartan's theorem).
- Integration step (constructing a 1 dimension bigger integral manifold) is done by appropriate 'Cauchy characteristics'. \square

Proposition 8.1.1. Let \mathcal{A} be a factor-algebra of $\Lambda_{\mathbb{R}}(C^\omega(\mathbb{R}^N))$, $\deg(\mathcal{A}) = n$, corresponding to a

$$\begin{array}{ccc} \mathcal{E}^q & \xrightarrow{\quad} & X \equiv b(F(\mathcal{A})) = = = \mathbf{Jet}^q(\mathbb{R}^{n+k}) \\ & \searrow & \downarrow \pi \\ & & \mathbb{R}^n \end{array} \quad , \dim(X) = N. \text{ Then}$$

system of differential equations

\mathcal{A} is in Cartan involution iff the following **Spencer δ -complex** is acyclic

$$0 \longrightarrow g^{(r)} \xrightarrow{\delta} g^{(r-1)} \otimes \Lambda^1(\mathbb{R}^n) \xrightarrow{\delta} g^{(r-2)} \otimes \Lambda^2(\mathbb{R}^n) \xrightarrow{\delta} \dots$$

$$\dots \xrightarrow{\delta} g^{(r-n)} \otimes \Lambda^n(\mathbb{R}^n) \longrightarrow 0$$

where $g^{(r)} := T(\mathbf{Jet}^r(\mathcal{E}^q)) \cap V\pi_{q+r-1}^{q+r} \hookrightarrow S_{q+r}(T^*\mathbb{R}^n) \otimes V\pi$ is r -th prolongation of symbol g , $\pi_{q+r-1}^{q+r} : \mathbf{Jet}^{q+r}(\mathbb{R}^{n+k}) \rightarrow \mathbf{Jet}^{q+r-1}(\mathbb{R}^{n+k})$ is a natural projection of jet-bundles, V is taking 'vertical' subbundle, S_p is p -th symmetric power,

$$\delta(\alpha_1 \cdots \alpha_{q+r-l} \otimes v \otimes \beta_1 \wedge \cdots \wedge \beta_l) := \sum_{i=1}^{q+r-l} \alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_{q+r-l} \otimes v \otimes \alpha_i \wedge \beta_1 \wedge \cdots \wedge \beta_l, \quad v \text{ is a section of } V\pi.$$

Proof. See [A-V-L, Sei, Vin2, V-K-L, Ver] □

Original Cartan's involutivity test was in terms of certain dimensions of 'infinitesimal integral elements'. The above theorem is due to J.P. Serre [A-V-L, La-Se].

9. Gelfand-Naimark 2-duality

Gelfand-Naimark duality is extendable to 2-duality over homotopies, which implies that cohomology theory for either C^* -algebras or compact Hausdorff spaces is automatically cohomology theory for the dual.

Let $\mathbf{C}^*\mathbf{Alg}^{\text{op}} \xrightleftharpoons[\text{G}]{\text{F}} \mathbf{CHTop}$ be the usual Gelfand-Naimark duality between commutative

C^* -algebras and compact Hausdorff spaces. Both categories are strict 2-categories with homotopy classes of homotopies as 2-cells (homotopy of C^* -algebras is a homotopy in \mathbf{Top} each instance of which is a C^* -algebra homomorphism). The reasonable question is: can it be extended to a 2-duality? The answer is yes.

By definition

$$\begin{array}{ccc} \mathbf{C}^*\mathbf{Alg}(A, B) \times |A| & \xrightarrow{ev} & |B| \\ f \times 1 \uparrow & \nearrow \bar{f} & \\ |I| \times |A| & & \end{array} \quad \begin{array}{ccc} \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A, B) & \xrightarrow{c_{A,B,\mathbb{C}}} & \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \\ 1 \times f \uparrow & \nearrow F(\bar{f}) & \\ \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| & & \end{array}$$

So that, if $f : |I| \times |A| \rightarrow |B|$ is a homotopy in $\mathbf{C}^*\mathbf{Alg}$, then its image in \mathbf{CHTop} is $F(\bar{f}) : |F(B)| \times |I| \rightarrow |F(A)|$ (where $| \cdot |$ denotes underlying set or map).

We need to prove that such extended F preserves 2-categorical structure (for G proof is symmetric).

Preserving homotopies

Lemma 9.1. *If B is locally compact then $\mathbf{Top}(B, C) \times \mathbf{Top}(A, B) \xrightarrow{c_{A,B,C}} \mathbf{Top}(A, C)$ is continuous (with compact-open topology in all hom-sets).*

Proof is standard. Let $f = g \circ h = c_{A,B,C}(g, h)$. Take U^K be a (subbase) nbhd of f . Sufficient to show that \exists (subbase) nbhds $U_1^{K^1} \ni g$, $U_2^{K^2} \ni h$, s.t. $U_1^{K^1} \circ U_2^{K^2} = c_{A,B,C}(U_1^{K^1}, U_2^{K^2}) \subset U^K$. Take $U_1 = U$, $K_2 = K$, K_1 be a compact nbhd of $h(K)$, s.t. $K_1 \subset g^{-1}(U)$ (K_1 exists by local compactness of B), $U_2 = \text{int}(K_1)$. □

Corollary. *If A is locally compact then $ev_{A,B} : \mathbf{Top}(A, B) \times |A| \rightarrow |B|$ is continuous.*

Proof. Each space A is homeomorphic to $\mathbf{Top}(1, A)$ (with compact-open topology), and $ev_{A,B}$ corresponds to $c_{1,A,B}$. \square

Lemma 9.2. • *Initial topology on $|F(A)| = \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C})$ w.r.t. evaluation maps $\forall a \in A$ $\mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \times 1 \xrightarrow{1 \times a} \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \times |A| \xrightarrow{ev} |\mathbb{C}|$ is point-open.*

• *Initial topology on $|G(X)| = \mathbf{CHTop}(X, \mathbb{C})$ w.r.t. evaluation maps $\forall x \in X$ $\mathbf{CHTop}(X, \mathbb{C}) \times 1 \xrightarrow{1 \times x} \mathbf{CHTop}(X, \mathbb{C}) \times |X| \xrightarrow{ev} |\mathbb{C}|$ is compact-open.*

Proof. See [P-Th], [Joh], [Eng]. \square

Lemma 9.3. *If $\mathcal{A}, \mathcal{B} \subset \mathbf{LCTop}$ are naturally dual subcategories of locally compact spaces (let D be a dualizing object) then if $\mathcal{A}(X, D)$ has compact-open topology (as initial topology w.r.t. evaluation maps) then initial topology of $|X| \cong \mathcal{B}(\mathcal{A}(X, D), D)$ is compact-open as well.*

Proof. Evaluation map $ev : \mathcal{A}(X, D) \times |X| \rightarrow |D|$ is continuous (since X is locally compact and $\mathcal{A}(X, D)$ has compact-open topology). It implies that initial (point-open) topology on $|X| \cong \mathcal{B}(\mathcal{A}(X, D), D)$ is actually compact-open [by assumption, topology of $|X|$ is initial w.r.t. all maps $'f' : |X| \xrightarrow{\sim} 1 \times |X| \xrightarrow{f \times 1} \mathcal{A}(X, D) \times |X| \xrightarrow{ev} |D|$. It means that topology on $|X| \cong \mathcal{B}(\mathcal{A}(X, D), D)$ is point-open since subbase open sets in point-open and initial topologies are the same $U'^{f'} := \{x \in |X| \mid 'f'(x) \in \bigcup_{open} U \subset D\} = 'f'^{-1}(U)$].

We need to show that $\{x \in |X| \mid \forall f \in \bigcup_{compact} K \subset \mathcal{A}(X, D). 'f'(x) \in \bigcup_{open} U \subset D\} = \bigcap_{f \in K} 'f'^{-1}(U)$

is open in point-open topology on $|X|$.

Take $x \in \bigcap_{f \in K} 'f'^{-1}(U)$, then $ev(K, x) \subset U$. By continuity of ev , $\forall y \in K. \exists V_y \ni y. \exists W_y \ni x$, s.t. $ev(V_y, W_y) \subset U$. $\bigcup_{y \in K} V_y \supset K$, so, by compactness, $\bigcup_{j=1, \dots, n} V_{y_j} \supset K$. Therefore, $ev(V_{y_j}, \bigcap_{j=1, \dots, n} W_{y_j}) \subset U$, $ev(\bigcup_{j=1, \dots, n} V_{y_j}, \bigcap_{j=1, \dots, n} W_{y_j}) \subset U$, $ev(K, \bigcap_{j=1, \dots, n} W_{y_j}) \subset U$, i.e., x is internal. \square

Corollary. *Gelfand-Naimark duality preserves homotopies.*

Proof. $|A| = \mathbf{CHTop}(X, \mathbb{C})$ has compact-open topology. $|X| = \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C})$ has point-open topology, so, by **Lemma 3** compact-open topology.

Multiplication $c_{A,B,\mathbb{C}}$ is continuous (since all hom-sets have compact-open topology). Therefore, $F(\bar{f})$ is continuous.

[In inverse direction $G : \mathbf{CHTop} \rightarrow \mathbf{C}^*\mathbf{Alg}$ there are no problem because $\mathbf{CHTop}(X, \mathbb{C})$ has compact-open topology. See also [Loo]]. \square

Preserving homotopy relation between homotopies

Let $\bar{f} : |I| \times |I| \times |A| \rightarrow |B|$ be continuous, s.t. $\bar{f}(0, t, a) = \bar{f}_0(t, a)$, $\bar{f}(1, t, a) = \bar{f}_1(t, a)$.

$$\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(A, B) \times |A| & \xrightarrow{ev} & |B| \\
\bar{f}^T \times 1_{|A|} \uparrow & \nearrow \bar{f} & \uparrow \\
|I| \times |I| \times |A| & & \bar{f}_0 \quad \bar{f}_1 \\
0 \times 1_{|I| \times |A|} \uparrow & \nearrow 1 \times 1_{|I| \times |A|} & \\
1 \times |I| \times |A| & & \\
<!, 1_{|I|} > \times 1_{|A|} \uparrow & \sim & \\
|I| \times |A| & &
\end{array}
\quad
\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A, B) & \xrightarrow{c_{A,B,\mathbb{C}}} & \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \\
1 \times \bar{f}^T \uparrow & \nearrow F(\bar{f}) & \uparrow \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| \times |I| & & F(\bar{f}_0) \quad F(\bar{f}_1) \\
1 \times ((0 \times 1_{|I|}) \circ <!, 1_{|I|} >) \uparrow & \nearrow 1 \times ((1 \times 1_{|I|}) \circ <!, 1_{|I|} >) & \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| & &
\end{array}$$

So, $F(\bar{f})$ is a homotopy from $F(\bar{f}_0)$ to $F(\bar{f}_1)$. $F(\bar{f})$ is continuous since $c_{A,B,\mathbb{C}}$ is continuous in compact-open topology. $\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C})$ has compact-open topology by **Lemma 4.1.3**.

Preserving unit 2-cells i_f

$$\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(A, B) \times |A| & \xrightarrow{ev} & |B| \\
f' \times 1 \uparrow & & \uparrow f \\
1 \times |A| & \xrightarrow{\sim} & |A| \\
! \times 1 \uparrow & \nearrow p_2 & \\
|I| \times |A| & &
\end{array}
\quad
\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A, B) & \xrightarrow{c_{A,B,\mathbb{C}}} & \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \\
1 \times f' \uparrow & \nearrow F(f \circ p_2) & \uparrow - \circ f \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times 1 & \xrightarrow{\sim} & \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \\
1 \times ! \uparrow & \nearrow F(i_f) & \uparrow p_1 \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| & &
\end{array}$$

So, if $i_f = f \circ p_2 \circ (! \times 1_{|A|}) = f \circ p_2$, then $F(i_f) = F(f) \circ p_1 = i_{F(f)}$.

Preserving composites $i_g * \bar{f} : |I| \times |A| \xrightarrow{\bar{f}} |B| \xrightarrow{g} |C|$
and $\bar{f} * i_h : |I| \times |A'| \xrightarrow{1 \times h} |I| \times |A| \xrightarrow{\bar{f}} |B|$

$$\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(A, C) \times |A| & \xrightarrow{ev} & |C| \\
(g \circ -) \times 1 \uparrow & & \uparrow g \\
\mathbf{C}^*\mathbf{Alg}(A, B) \times |A| & \xrightarrow{ev} & |B| \\
f \times 1 \uparrow & \nearrow \bar{f} & \\
|I| \times |A| & &
\end{array}
\quad
\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(C, \mathbb{C}) \times |I| & & \\
1 \times (\mathbf{C}^*\mathbf{Alg}(A, g) \circ f) \downarrow & \nearrow F(g \circ \bar{f}) & \\
\mathbf{C}^*\mathbf{Alg}(C, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A, C) & \xrightarrow{c_{A,C,\mathbb{C}}} & \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \\
\mathbf{C}^*\mathbf{Alg}(g, \mathbb{C}) \times 1 & & \uparrow \sim \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A, B) & \xrightarrow{c_{A,B,\mathbb{C}}} & \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \\
1 \times f \uparrow & \nearrow F(\bar{f}) & \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| & &
\end{array}$$

$g \circ \bar{f}$ is a homotopy corresponding to $\mathbf{C}^*\mathbf{Alg}(A, g) \circ f$. Outer perimeter of the right diagram commutes because of definition of $F(\bar{f})$, $F(g \circ \bar{f})$ and associativity law [if $(s, t) \in \mathbf{C}^*\mathbf{Alg}(C, \mathbb{C}) \times |I|$ then $s \circ (g \circ f(t)) = (s \circ g) \circ f(t)$]. So, $F(g \circ \bar{f}) = F(\bar{f}) \circ (F(g) \times 1_{|I|})$, i.e., $F(i_g * \bar{f}) = F(\bar{f}) * i_{F(g)}$.

$$\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(A, B) \times |A| & \xrightarrow{ev} & |B| \\
f \times 1 \uparrow & \nearrow \bar{f} & \uparrow \\
|I| \times |A| & & \bar{f} \circ (1 \times h) \\
1 \times h \uparrow & & \uparrow \\
|I| \times |A'| & & 1 \sim \\
(\bar{f} \circ (1 \times h))^T \times 1 \downarrow & \searrow \bar{f} \circ (1 \times h) = ev \circ (f \times 1) \circ (1 \times h) = ev \circ (f \times h) & \downarrow \\
\mathbf{C}^*\mathbf{Alg}(A', B) \times |A'| & \xrightarrow{ev} & |B|
\end{array}
\quad
\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A, B) & \xrightarrow{c_{A, B, \mathbb{C}}} & \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \\
1 \times f \uparrow & \nearrow F(\bar{f}) & \downarrow \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| & & \mathbf{C}^*\mathbf{Alg}(h, \mathbb{C}) \\
1 \times (\bar{f} \circ (1 \times h))^T \downarrow & \searrow F(\bar{f} \circ (1 \times h)) & \downarrow \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A', B) & \xrightarrow{c_{A', B, \mathbb{C}}} & \mathbf{C}^*\mathbf{Alg}(A', \mathbb{C})
\end{array}$$

Right internal triangle of the right diagram commutes since if $(g, t) \in \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I|$ then $\mathbf{C}^*\mathbf{Alg}(h, \mathbb{C}) \circ c_{A, B, \mathbb{C}} \circ (1 \times f)(g, t) = (g \circ f(t)) \circ h = g \circ (f(t) \circ h) = c_{A', B, \mathbb{C}}(g, f(t) \circ h) = c_{A', B, \mathbb{C}}(g, (\bar{f} \circ (1 \times h))^T(t)) = c_{A', B, \mathbb{C}} \circ (1 \times (\bar{f} \circ (1 \times h))^T)(g, t)$. So, $F(\bar{f} * i_h) = F(\bar{f} \circ (1 \times h)) = F(h) \circ F(\bar{f}) = i_{F(h)} * F(\bar{f})$.

Preserving vertical composites

We need to show if $\bar{f} : \bar{f} \circ i_0 \simeq \bar{f} \circ i_1$ and $\bar{g} : \bar{g} \circ i_0 \simeq \bar{g} \circ i_1$ are homotopies in $\mathbf{C}^*\mathbf{Alg}$ s.t. $\bar{f} \circ i_1 = \bar{g} \circ i_0$ then $F(\bar{g} \odot \bar{f}) = F(\bar{g}) \odot F(\bar{f})$.

By definition, vertical composite $\bar{g} \odot \bar{f}$ is

$$\begin{array}{ccc}
|A| \times |[0, \frac{1}{2}]| & \xleftarrow[\sim]{1 \times \alpha} & |A| \times |I| \\
& \searrow 1 \times i & \nearrow \bar{f} \\
& & |A| \times |I| \xrightarrow[\sim]{\exists! \bar{g} \odot \bar{f}} |B| \\
& \swarrow 1 \times j & \nearrow \bar{g} \\
|A| \times |[\frac{1}{2}, 1]| & \xleftarrow[\sim]{1 \times \beta} & |A| \times |I|
\end{array}$$

$$\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(A, B) \times |A| & \xrightarrow{ev} & |B| \\
(\bar{g} \odot \bar{f})^T \times 1 \uparrow & \nearrow \bar{g} \odot \bar{f} & \uparrow \\
|I| \times |A| & & \bar{f} \\
j \times 1 \uparrow & \nearrow i \times 1 & \uparrow \\
|[\frac{1}{2}, 1]| \times |A| & & |[0, \frac{1}{2}]| \times |A| \\
f \times 1 \uparrow & \nearrow \alpha \times 1 & \uparrow \\
|I| \times |A| & & |I| \times |A| \\
\beta \times 1 \uparrow & \nearrow g \times 1 & \uparrow
\end{array}
\quad
\begin{array}{ccc}
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times \mathbf{C}^*\mathbf{Alg}(A, B) & \xrightarrow{c_{A, B, \mathbb{C}}} & \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C}) \\
1 \times (\bar{g} \odot \bar{f})^T \uparrow & \nearrow F(\bar{g} \odot \bar{f}) & \downarrow \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| & & \mathbf{C}^*\mathbf{Alg}(h, \mathbb{C}) \\
1 \times (j \odot \beta) \uparrow & \nearrow F(\bar{g}) & \downarrow \\
\mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| & & \mathbf{C}^*\mathbf{Alg}(B, \mathbb{C}) \times |I| \\
& \searrow 1 \times (i \odot \alpha) & \downarrow \\
& & \mathbf{C}^*\mathbf{Alg}(A, \mathbb{C})
\end{array}$$

By uniqueness $f \equiv \bar{f}^T = (\bar{g} \odot \bar{f})^T \circ i \circ \alpha$, $g \equiv \bar{g}^T = (\bar{g} \odot \bar{f})^T \circ j \circ \beta$.

So, $\begin{cases} F(\bar{g} \odot \bar{f}) \odot (1 \times (i \circ \alpha)) = F(\bar{f}) \\ F(\bar{g} \odot \bar{f}) \odot (1 \times (j \circ \beta)) = F(\bar{g}) \end{cases}$. It means $F(\bar{g} \odot \bar{f}) = F(\bar{g}) \odot F(\bar{f})$.

Preserving horisontal composites $A \begin{array}{c} \xrightarrow{f_0} \\ \Downarrow \bar{f} \\ \xrightarrow{f_1} \end{array} B \begin{array}{c} \xrightarrow{g_0} \\ \Downarrow \bar{g} \\ \xrightarrow{g_1} \end{array} C$

$\bar{g} * \bar{f} := (\bar{g} * i_{f_1}) \odot (i_{g_0} * \bar{f}) \simeq (i_{g_1} * \bar{f}) \odot (\bar{g} * i_{f_0})$ (homotopic homotopies).
 $F(\bar{g} * \bar{f}) = F(\bar{g} * i_{f_1}) \odot F(i_{g_0} * \bar{f}) = (i_{F(f_1)} * F(\bar{g})) \odot (F(\bar{f}) * i_{F(g_0)}) \simeq F(\bar{f}) * F(\bar{g})$.

Proposition 9.1 completes the proof of Gelfand-Naimark 2-duality $\mathbf{C}^* \mathbf{Alg}^{\text{op}} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathbf{CHTop}$.

Proposition 9.1. If $\mathbf{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathbf{D}$ are two strict n -categories and two strict n -functors in the op-

posite directions such that the restriction $\mathbf{C}^{\leq 1} \begin{array}{c} \xrightarrow{F^{\leq 1}} \\ \perp \\ \xleftarrow{G^{\leq 1}} \end{array} \mathbf{D}^{\leq 1}$ is an adjunction with unit $\eta : 1_{\mathbf{C}^{\leq 1}} \rightarrow$

$G^{\leq 1} F^{\leq 1}$ and counit $\varepsilon : F^{\leq 1} G^{\leq 1} \rightarrow 1_{\mathbf{D}^{\leq 1}}$ which are still natural transformations for the extension (i.e. $\eta : 1_{\mathbf{C}} \rightarrow GF$ and $\varepsilon : FG \rightarrow 1_{\mathbf{D}}$ are natural transformations) then the extended situation

$\mathbf{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathbf{D}$ is a strict adjunction.

Proof. A strict adjunction is completely determined by its 'unit-counit' (proposition 1.5.3). $\eta : 1_{\mathbf{C}} \rightarrow GF$ and $\varepsilon : FG \rightarrow 1_{\mathbf{D}}$ are natural transformations and satisfy triangle identities $\varepsilon F \circ_1 F \eta = 1_F$ and $G \varepsilon \circ_1 \eta G = 1_G$ (because, e.g. $\varepsilon F = \varepsilon F^{\leq 1}$, $1_F = 1_{F^{\leq 1}}$ (set-theoretically), etc.) \square

Corollary. Any 1-adjunction between a category of topological algebras and a subcategory of topological spaces is a 2-adjunction if it can be extended functorially over 2-cells in the way that each instance of the image of a homotopy is the image of this instance of the preimage-homotopy.

Proof. Under given conditions unit and counit of 1-adjunction are automatically natural trans-

formations for the extension. E.g., take unit η . Naturality square $\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ f^1 \downarrow & & \downarrow GF f^1 \\ B & \xrightarrow{\eta_B} & GFB \end{array}$, where

$f^1 : A \times I \rightarrow B$ is a homotopy, holds because each instance of it holds (since η is a unit of 1-adjunction), i.e. $\forall t \in I \ \eta_B \circ f^1(-, t) = GF(f^1(-, t)) \circ \eta_A$, it means $\eta_B \circ f^1 = GF(f^1) \circ (\eta_A \times I)$, i.e. $\eta_B * f^1 = GF(f^1) * \eta_A$. \square

Gelfand-Naimark case is one of the above corollary. End of proof of Gelfand-Naimark 2-duality. \square

Remark. There are 'forgetful' functors $\mathbf{C}^* \mathbf{Alg} \rightarrow 2\text{-Set}$ and $\mathbf{CHTop} \rightarrow 2\text{-Set}$ (where 2-Set is the usual \mathbf{Set} with just one iso-2-cell for each pair of maps with the same domain and codomain) but they are not faithful and forget too much in order 2-Set could be an underlying category of Gelfand-Naimark 2-duality. \square

Proposition 9.2. • *Gelfand-Naimark 2-duality is concrete over 2-Cat (2-Cat is the usual 2-category of (small) categories, functors and natural transformations), i.e. \exists (faithful) forgetful*

functors $U : \mathbf{C}^\mathbf{Alg} \rightarrow 2\text{-Cat}$ and $V : \mathbf{CHTop} \rightarrow 2\text{-Cat}$ such that*

$$\begin{array}{ccc} \mathbf{C}^*\mathbf{Alg}^{\text{op}} & \xrightarrow{F} & \mathbf{CHTop} \\ & \searrow \mathbf{C}^*\mathbf{Alg}(-, \mathbb{C}) & \downarrow V \\ & & 2\text{-Cat} \end{array} \quad \text{and}$$

$$\begin{array}{ccc} \mathbf{CHTop}^{\text{op}} & \xrightarrow{G^{\text{op}}} & \mathbf{C}^*\mathbf{Alg} \\ & \searrow \mathbf{CHTop}(-, \mathbb{C}) & \downarrow U \\ & & 2\text{-Cat} \end{array} \quad \text{where } U \text{ and } V \text{ are composites of inclusion and fundamental groupoid}$$

functors ($U : \mathbf{C}^\mathbf{Alg} \hookrightarrow 2\text{-Top} \xrightarrow{2\text{-Top}(1, -)} 2\text{-Cat}$ and $V : \mathbf{CHTop} \hookrightarrow 2\text{-Top} \xrightarrow{2\text{-Top}(1, -)} 2\text{-Cat}$).*

- *This duality is natural, i.e. lifting of hom-functors $\mathbf{C}^*\mathbf{Alg}(-, \mathbb{C})$, $\mathbf{CHTop}(-, \mathbb{C})$ along V and U is initial.* \square

Remark. 2-duality allows us to transfer (co)homology theories from one side to another. Under a reasonable assumption that K-theory was determined in a universal way we could get **M. Atiyah theorem** that *K-groups of C^* -algebras and compact Hausdorff spaces coincide*. The problem, however, is that K-groups were determined technically (not universally). But, there is a theorem by J. Cuntz [Weg] that K-theory is universally determined on a large subcategory of C^* -algebras. \square

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